

# Three particles in a box: Mapping the finite-volume spectrum to the S-matrix

Maxwell T. Hansen

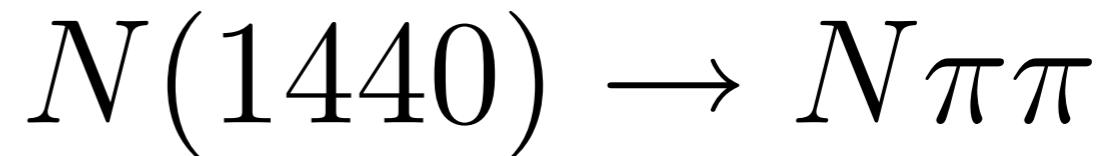
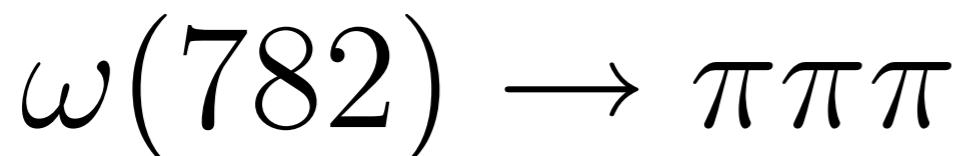
Institut für Kernphysik and HIM, JG-Universität Mainz

February 5th, 2014

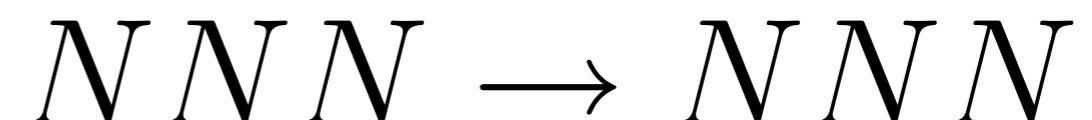
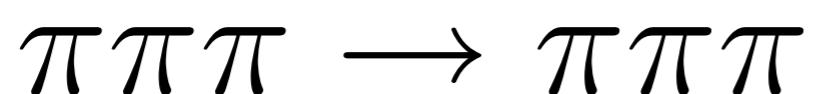
*MTH and Stephen R. Sharpe, arXiv:1408.5933, 2014  
(published in PRD)*

The three-particle quantization condition  
is a necessary first step for using LQCD  
to investigate...

**resonances decaying to three or more hadrons**



**three-body forces**



The three-particle quantization condition  
is a necessary first step for using LQCD  
to investigate...

**weak decays coupling to three or more hadrons**

$$K \rightarrow \pi\pi\pi$$

$$D \rightarrow \pi\pi \quad D \rightarrow K \bar{K}$$

(couples to  $\pi\pi\pi\pi$ )

---

Need QCD scattering amplitudes to relate  
finite-volume lattice matrix elements to  
physical decay amplitudes

Lellouch, L. & Lüscher, M.  
*Commun. Math. Phys.* 219, 31-44 (2001)

# Outline

Two particle review

Three particle quantization condition

Relation to scattering amplitude

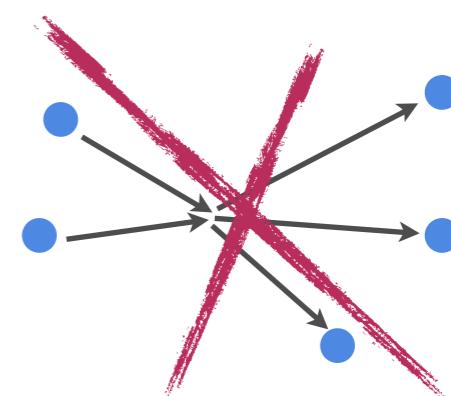
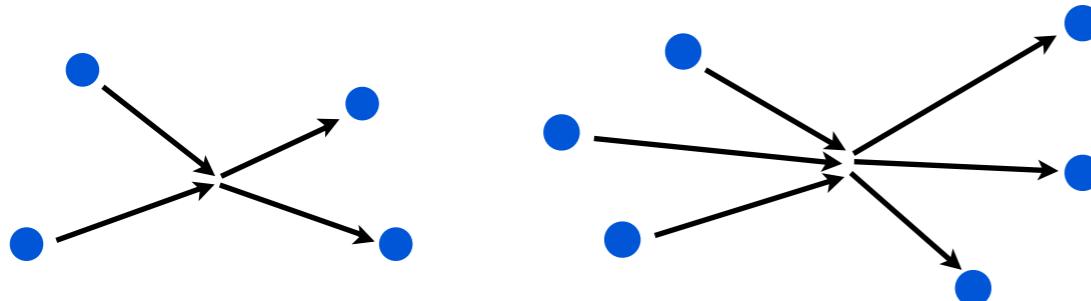
Conclusion

# Particle content

Single scalar, mass  $m$  ←  
all results for  
identical scalars

Relativistic field theory

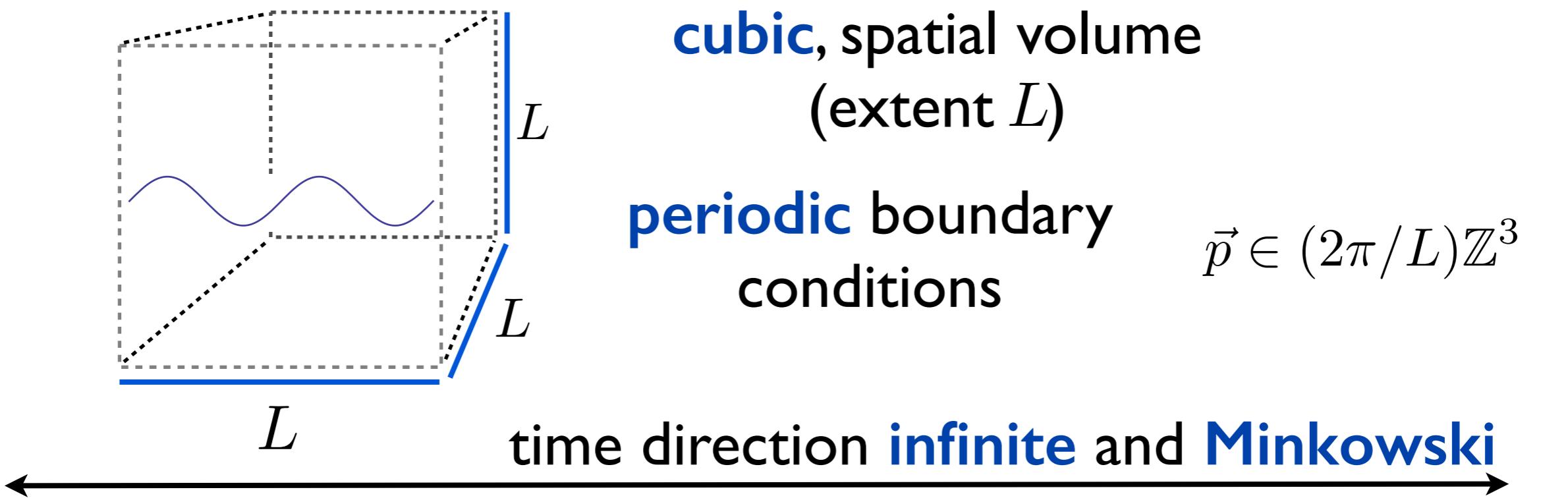
$\mathbb{Z}_2$  symmetry



(For pions in QCD this is G-parity)

**Include all vertices  
with even number of legs**

# Finite volume



Take  $L$  large enough to ignore  $e^{-mL}$

dropped throughout!

Take space to be continuous

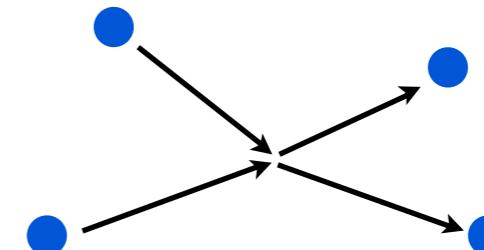
lattice spacing set  
to zero

# Two particles in a box

$$4m \quad \overbrace{\hspace{1cm}}^{\text{---}} \quad E_2^*(L, \vec{P})$$
$$\overbrace{\hspace{1cm}}^{\text{---}} \quad E_1^*(L, \vec{P})$$
$$\overbrace{\hspace{1cm}}^{\text{---}} \quad E_0^*(L, \vec{P})$$



$$i\mathcal{M}_{2 \rightarrow 2}$$



$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T \sigma(x) \sigma^\dagger(0) | 0 \rangle$$

Require  $E^* < 4m$

**even-particle quantum numbers**

Calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

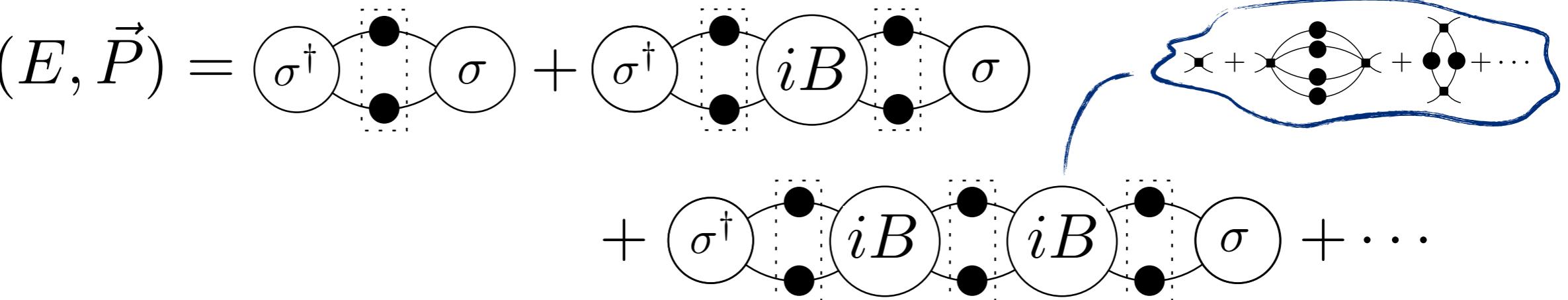
Following Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

# Two particles in a box

1

$$C_L(E, \vec{P}) = \langle \sigma^\dagger \rangle + \langle \sigma^\dagger iB \rangle + \langle iB iB \rangle + \dots$$

$\quad + \langle \sigma^\dagger iB \rangle + \dots$



# Two particles in a box

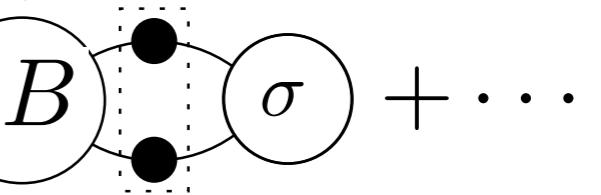
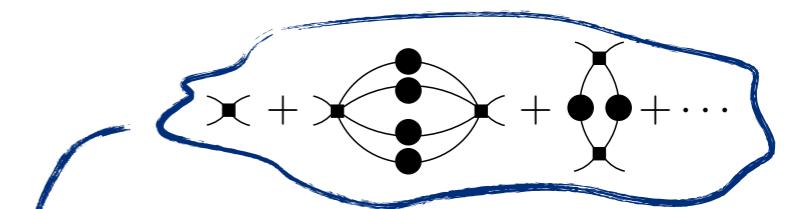
$$C_L(E, \vec{P}) = \langle \sigma^\dagger \rangle \langle \sigma \rangle + \langle \sigma^\dagger \rangle \langle iB \rangle \langle \sigma \rangle$$

1

$$\langle \sigma^\dagger \rangle \langle \sigma \rangle + \langle \sigma^\dagger \rangle \langle \sigma \rangle$$

2

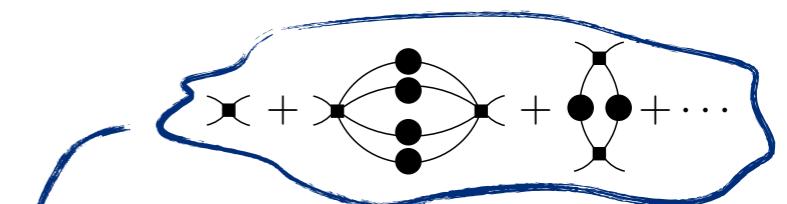
$$+ \langle \sigma^\dagger \rangle \langle iB \rangle \langle iB \rangle \langle \sigma \rangle + \dots$$



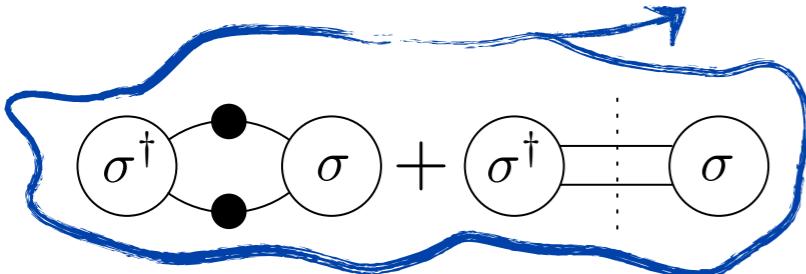
# Two particles in a box

$$C_L(E, \vec{P}) = \langle \sigma^\dagger \rangle \langle \sigma \rangle + \langle \sigma^\dagger \rangle \langle iB \rangle \langle \sigma \rangle$$

1



2



$$+ \langle \sigma^\dagger \rangle \langle iB \rangle \langle iB \rangle \langle \sigma \rangle + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

$$+ \langle A \rangle \langle A' \rangle + \langle A \rangle \langle i\mathcal{M} \rangle \langle A' \rangle$$

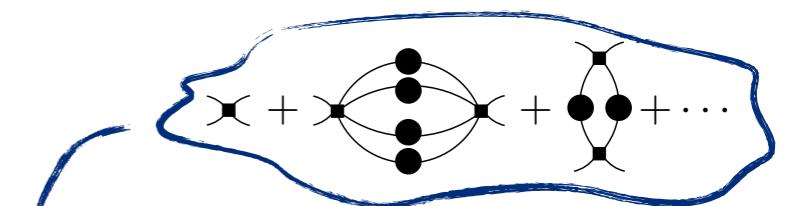
3

$$+ \langle A \rangle \langle i\mathcal{M} \rangle \langle i\mathcal{M} \rangle \langle A' \rangle + \dots$$

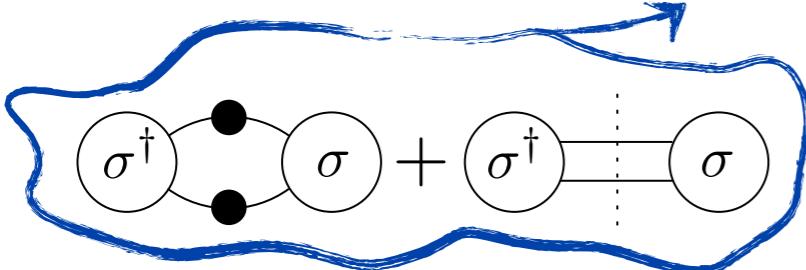
# Two particles in a box

$$C_L(E, \vec{P}) = \langle \sigma^\dagger \rangle \langle \sigma \rangle + \langle \sigma^\dagger \rangle \langle iB \rangle \langle \sigma \rangle$$

1



2



$$+ \langle \sigma^\dagger \rangle \langle iB \rangle \langle iB \rangle \langle \sigma \rangle + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

$$+ \langle A \rangle \langle A' \rangle + \langle A \rangle \langle i\mathcal{M} \rangle \langle A' \rangle$$

3

$$+ \langle A \rangle \langle i\mathcal{M} \rangle \langle i\mathcal{M} \rangle \langle A' \rangle + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

4

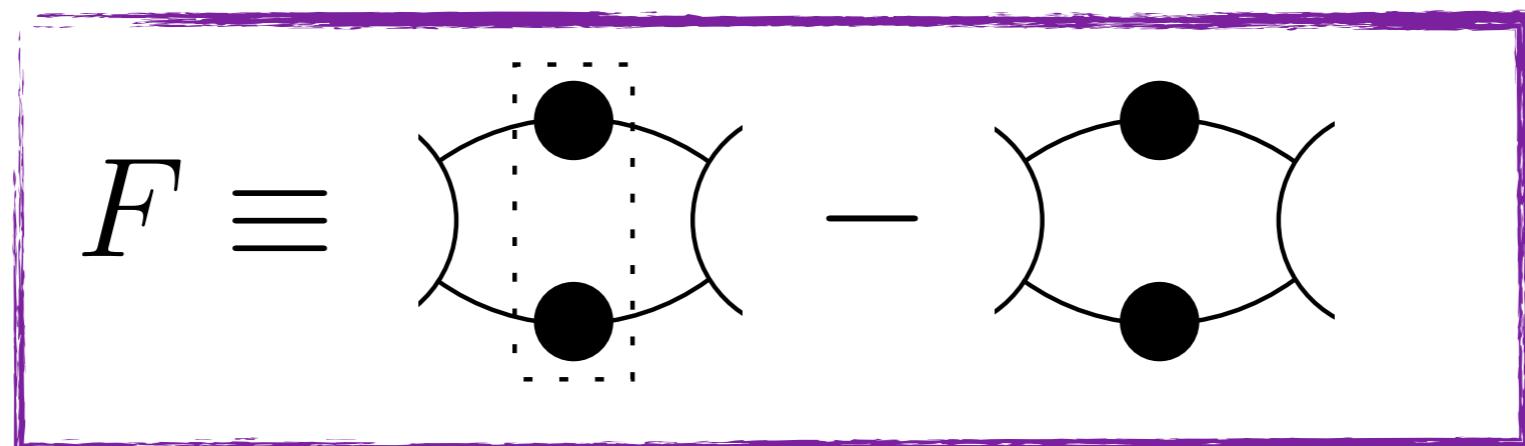
# Two-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum  
is all solutions to

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2 \rightarrow 2} iF] = 0$$

diagonal matrix in  
angular momentum space

kinematic  
(related to Lüscher  
Zeta function)



# Two-particle result

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2\rightarrow 2}iF] = 0$$

...is it useful?

**At low energies, s-wave dominates**

$$[\mathcal{M}_{2\rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

$$F^s(E, \vec{P}, L) \equiv \frac{1}{2} \left[ \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right] \frac{1}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k} + i\epsilon)}$$

# Two-particle result

**Note also, equation is real**

$$[\mathcal{M}_{2 \rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

$$p_n \cot \delta^s(p_n) - i \cancel{p}_n = -16\pi E_n^* \operatorname{Re} F^s - i \cancel{p}_n$$

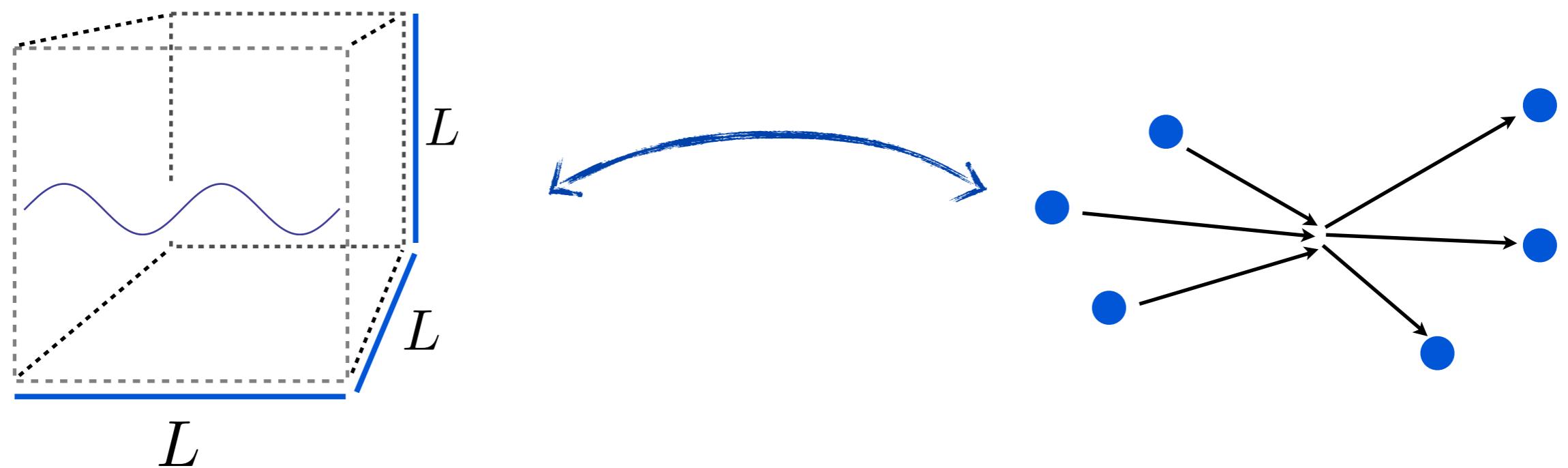
This can also be seen by replacing i-epsilon with principal value everywhere in derivation.

$$\mathcal{M}_{2 \rightarrow 2} \longrightarrow \mathcal{K}_{2 \rightarrow 2}$$

$$F \longrightarrow \operatorname{Re} F$$

**Important for three-particle case**

# Now, three particles in a box

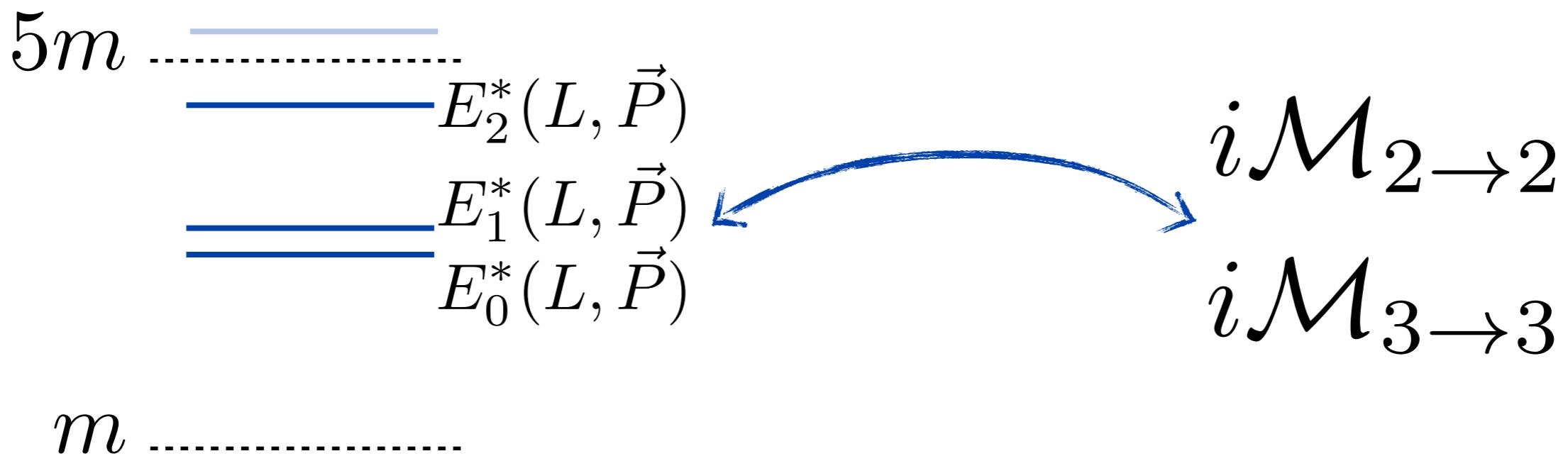


# Three particles in a box

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

Require  $m < E^* < 5m$

**odd-particle quantum numbers**



**Assume no two-particle bound state**

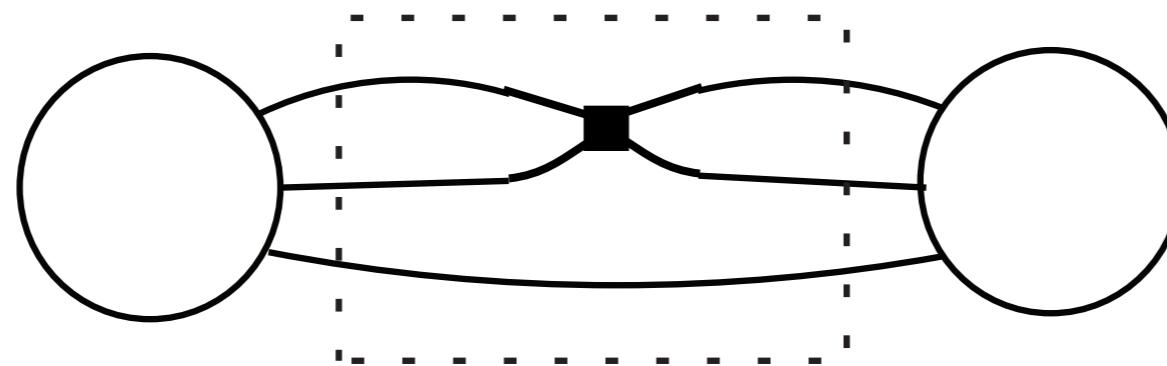
# New skeleton expansion

$$C_L(E, \vec{P}) = ? = \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots$$

The diagram consists of three horizontal rows of circles. The top row has two white circles connected by a double line. The middle row has two white circles connected by a double line. The bottom row has two white circles connected by a double line. Vertical dashed lines connect the top circle of one row to the middle circle of the next row, and the middle circle of one row to the bottom circle of the next row. This creates a ladder-like structure between the rows.

## (propagators still fully dressed)

# No! We also need diagrams like



(  **should only contain connected diagrams**)

# New skeleton expansion

$$C_L(E, \vec{P}) = \text{(Diagram with orange circles)} + \text{(Diagram with purple circles)} + \dots$$

The equation shows the function  $C_L(E, \vec{P})$  as a sum of two types of diagrams. The first type, represented by orange circles, has three horizontal lines entering and exiting each circle. The second type, represented by purple circles, has four horizontal lines entering and exiting each circle. Dashed boxes group the first few terms of each type, followed by a plus sign and ellipses.

## Kernel definitions:

$$\text{(Diagram with purple circle)} \equiv \text{(Diagram with black dot)} + \text{(Diagram with two horizontal lines and a central loop)} + \text{(Diagram with a vertical loop)} + \dots$$

$$\text{(Diagram with orange circle)} \equiv \text{(Diagram with black dot)} + \text{(Diagram with a horizontal line and a central dot)} + \text{(Diagram with a horizontal line and a central loop)} + \dots$$

# New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$
$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$
$$+ \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots$$
$$+ \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \dots$$

## Kernel definitions:

$$\text{Diagram 1} \equiv \text{x} + \text{x} + \text{x} + \dots$$

$$\text{Diagram 2} \equiv \text{x} + \text{x} + \text{x} + \dots$$

# New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$
$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$
$$+ \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots$$
$$+ \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \dots$$
$$+ \dots$$
$$+ \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} + \dots$$

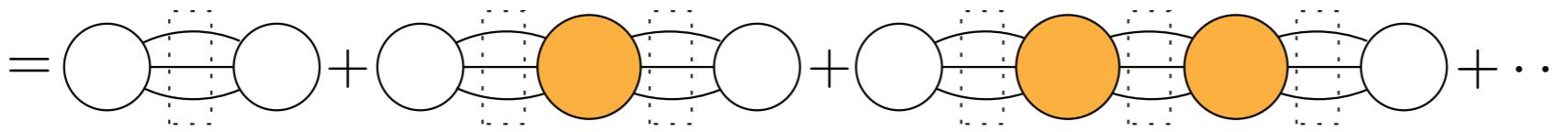
**Kernel definitions:**

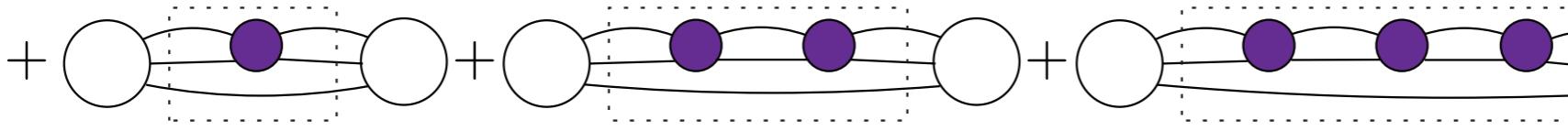
$$\text{Diagram 1} \equiv \text{x} + \text{x} + \text{x} + \dots$$

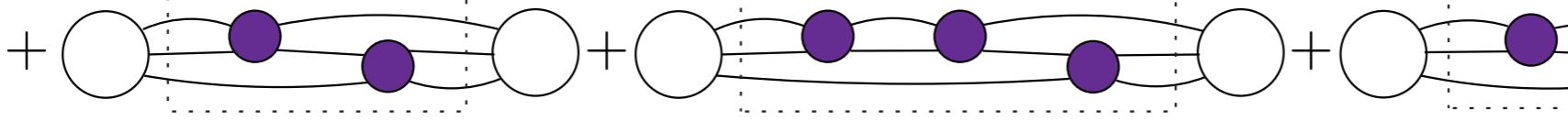
$$\text{Diagram 2} \equiv \text{x} + \text{x} + \text{x} + \dots$$

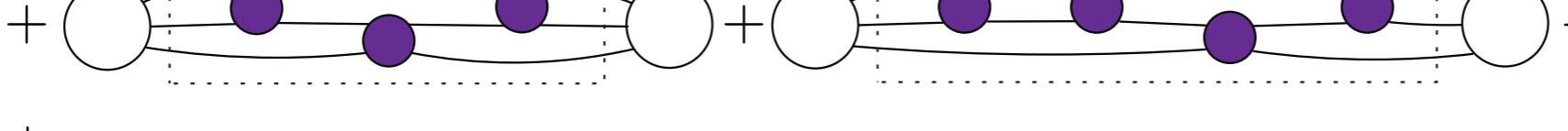
# New skeleton expansion

$$C_L(E, \vec{P}) = \dots + \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

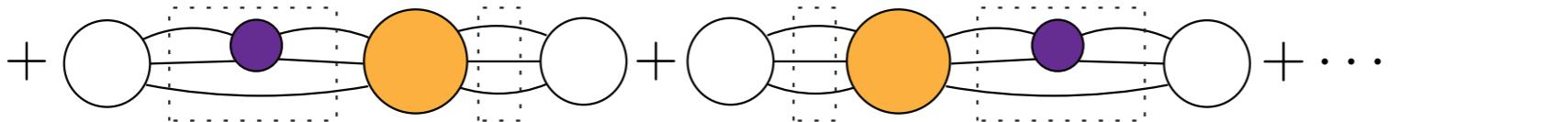
+ 

+ 

+ 

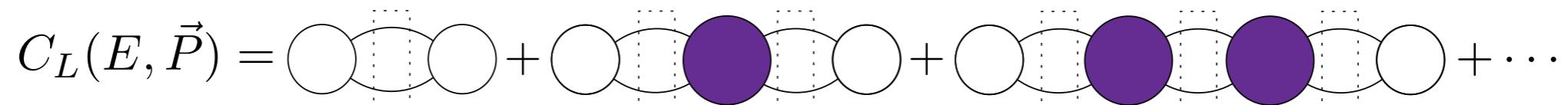
+ 

+  $\dots$

+ 

**Compare to two-particle skeleton expansion**

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$



# What is new here?

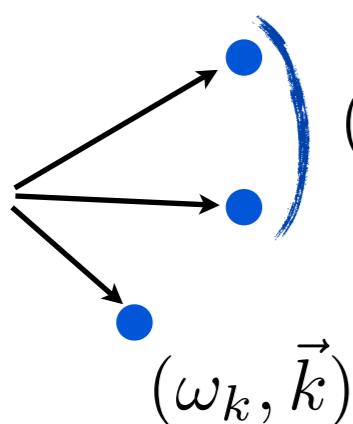
1. Degrees of freedom are different

two particles

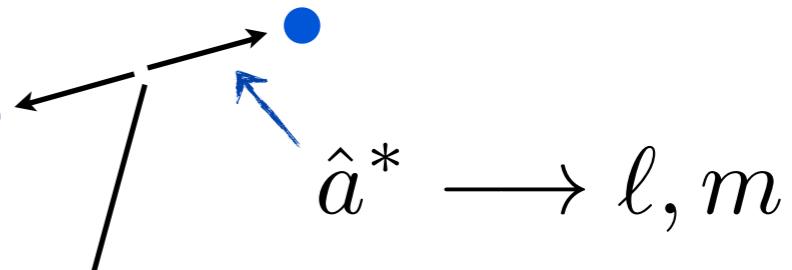
two-particle angular  
momentum

three particles

$\vec{k}$  + two-particle angular  
momentum



BOOST



Our result only depends  
on finite-volume  $\vec{k}$

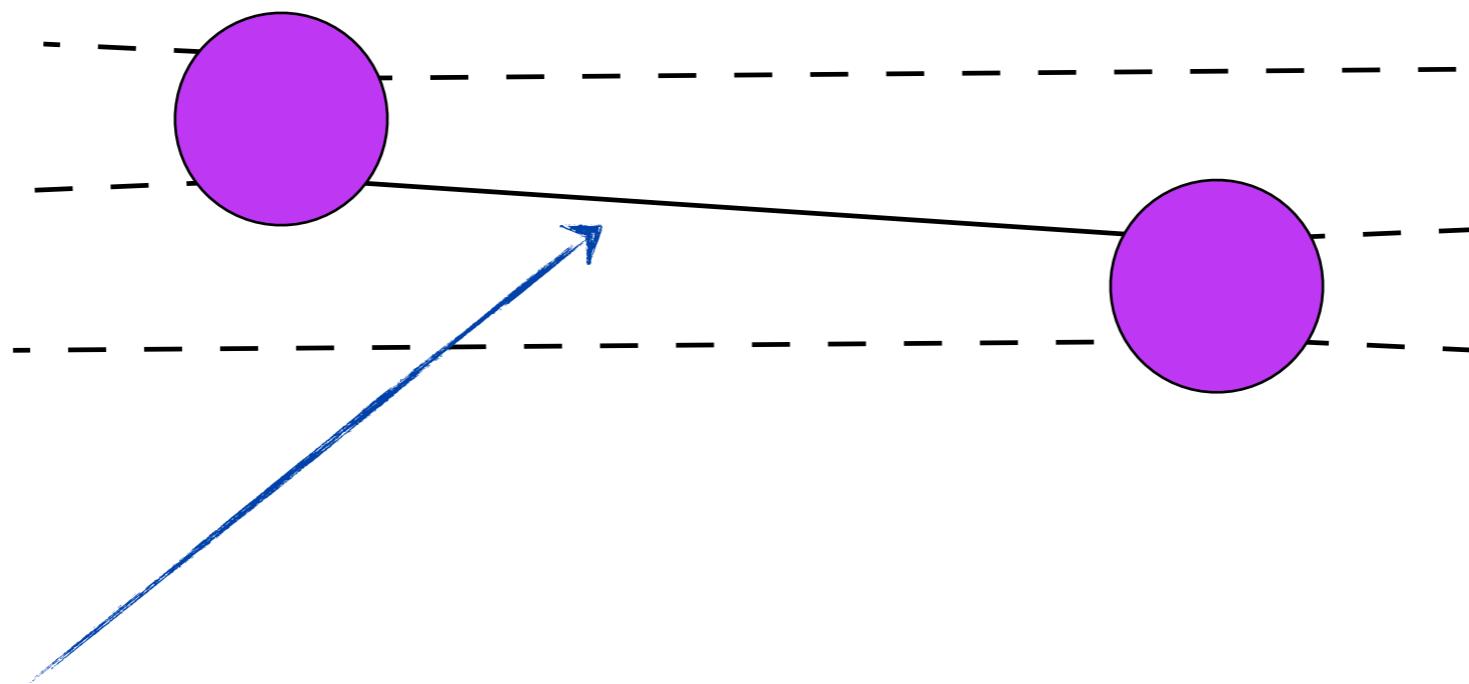
$$\vec{k} = \frac{2\pi}{L} \vec{n}$$

integer vector

# What is new here?

## 2. Three particle divergences

Three particle observable  $i\mathcal{M}_{3 \rightarrow 3}$   
contains the diagram



Certain external momenta  
put this on-shell

$i\mathcal{M}_{3 \rightarrow 3}$   
has singularities

# What is new here?

## 2. Three particle divergences

Define  $i\mathcal{M}_{\text{df}, 3 \rightarrow 3}$

$$\equiv i\mathcal{M}_{3 \rightarrow 3} - \left[ i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} + \int i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} + \dots \right]$$

The diagram shows a three-particle scattering process. On the left, two particles (represented by purple circles) enter from below and interact via a central vertex to produce a single outgoing particle (purple circle). A vertical dashed line labeled  $S$  passes through the central vertex. On the right, a similar process is shown, but it includes a loop diagram where the outgoing particle from the first interaction becomes an incoming particle for a second interaction with another particle, also passing through the same vertical line  $S$ . This represents the subtraction term in the definition of the three-particle divergence.

only on-shell amplitudes here

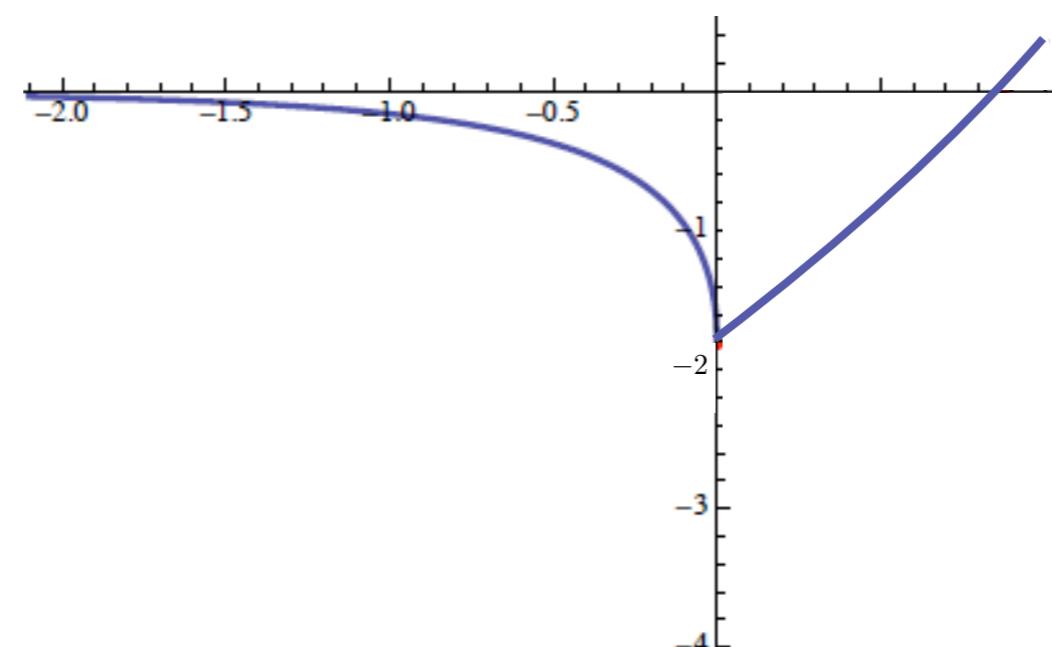
infinite series  
built with factors of  $S i\mathcal{M}_{2 \rightarrow 2}$

This subtraction emerges naturally in our finite-volume analysis

# What is new here?

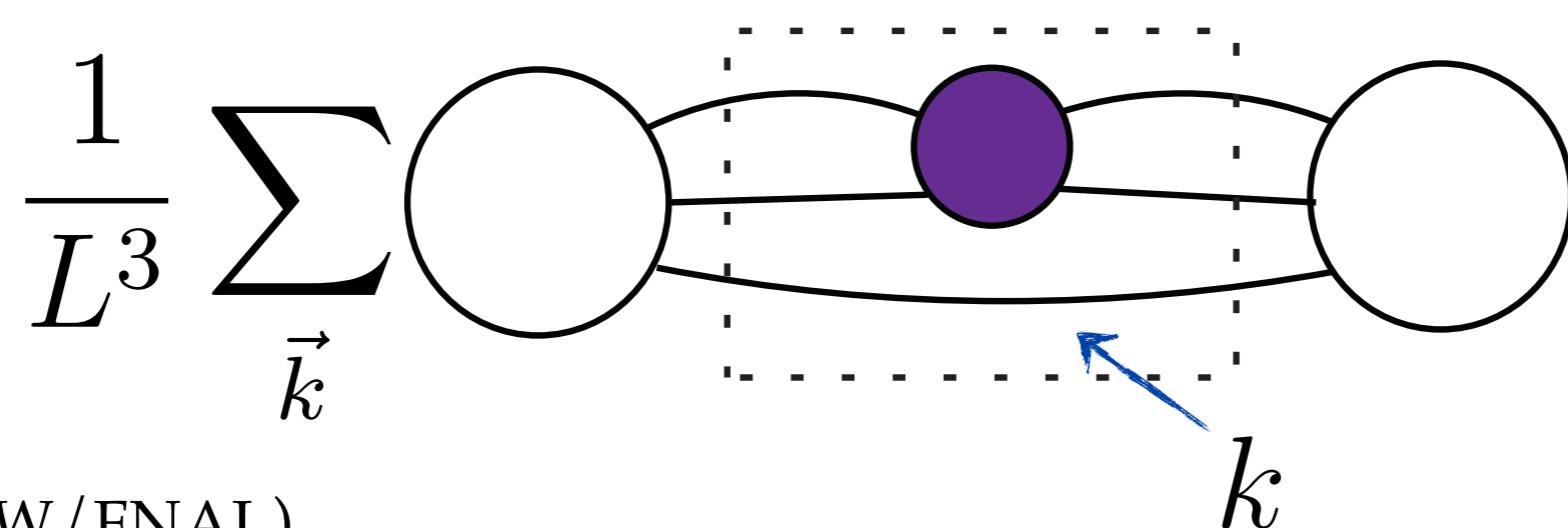
3. Must now worry about sum crossing  
two-particle unitary cusp

two-particle  
scattering  
(real part)



depends on  $k$

two particle energy



# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

To remove cusp

$i\epsilon$  prescription



principal  
value  $\widetilde{PV}$

Analytically continue principal value below threshold  
then interpolate to prescription-free subthreshold form

Polejaeva, K. and Rusetsky, A. *Eur. Phys. J.* A48 (2012) 67

# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

To remove cusp

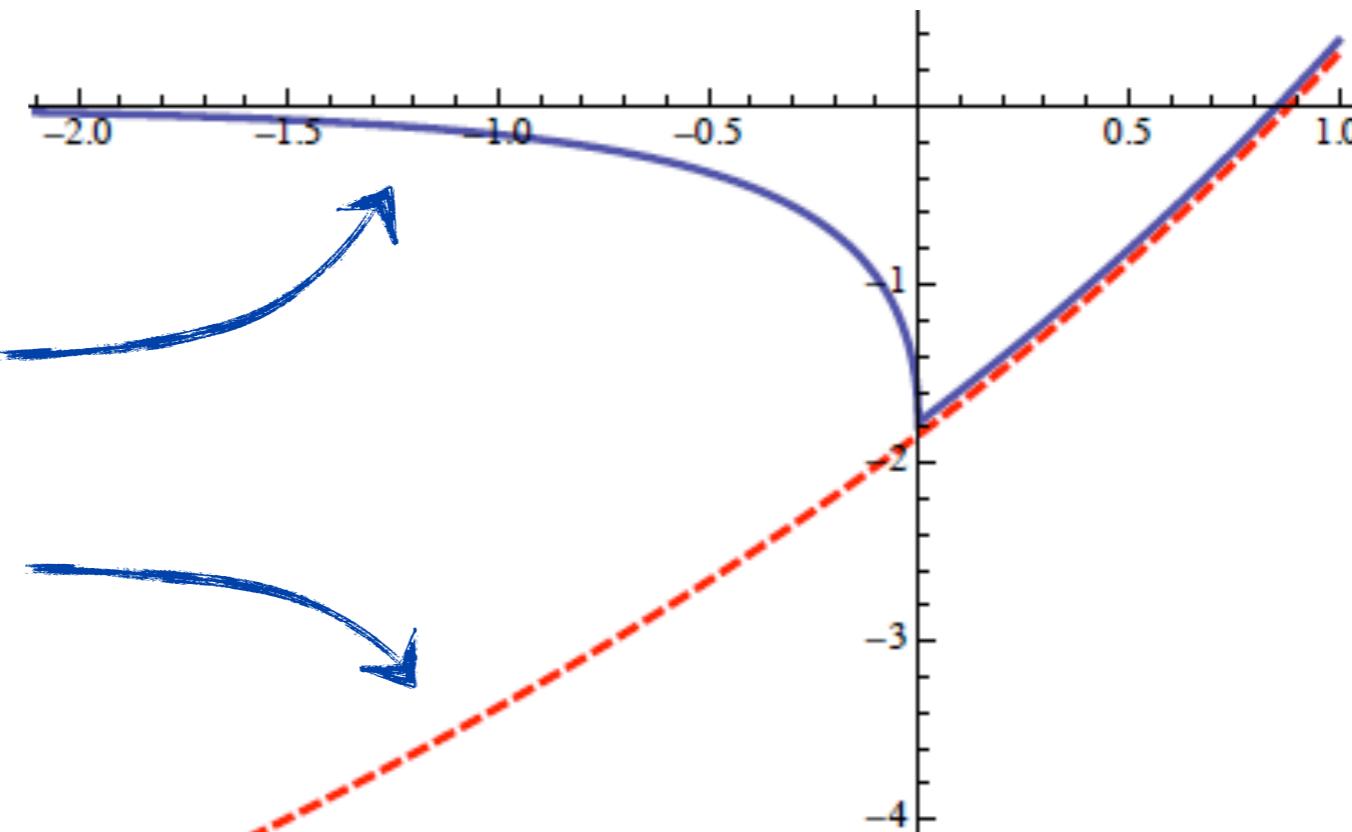
$i\epsilon$  prescription

principal value  $\widetilde{PV}$

standard definition

modification

$\widetilde{PV}$



# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

$$i\mathcal{M}_{2 \rightarrow 2} = \langle \dots \rangle + \langle \dots \rangle + \langle \dots \rangle + \dots$$

has cusp

$$i\mathcal{K}_{2 \rightarrow 2} = \text{[purple blob with 4 black legs]} =$$

$$\langle \dots \rangle + \langle \dots \rangle + \langle \dots \rangle + \dots$$

has no cusp

# What is new here?

3. Must now worry about sum crossing  
two-particle unitary cusp

$$\begin{array}{ccc} i\mathcal{M}_{2 \rightarrow 2} & \xrightarrow{\hspace{2cm}} & i\mathcal{K}_{2 \rightarrow 2} \\ i\mathcal{M}_{\text{df}, 3 \rightarrow 3} & & i\mathcal{K}_{\text{df}, 3 \rightarrow 3} \end{array}$$

**We relate these infinite-volume quantities  
to the finite-volume spectrum**

# Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df}, 3 \rightarrow 3} iF_3} A_3$$

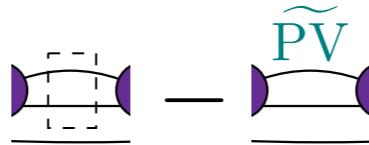
$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L, 2 \rightarrow 2} iG} i\mathcal{M}_{L, 2 \rightarrow 2} iF \right]$$

$$i\mathcal{M}_{L, 2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

# Three-particle result

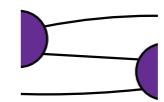
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{df,3 \rightarrow 3} iF_3} A_3$$

**sum-integral  
difference**



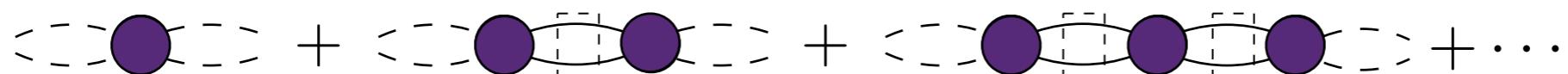
$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right]$$

**encodes  
switches**



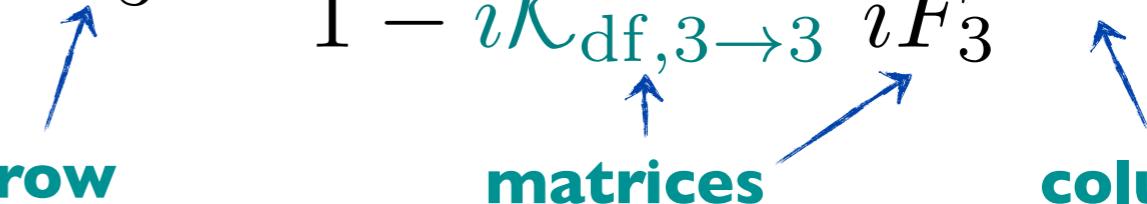
$$i\mathcal{M}_{L,2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

**sum of all two-particle loops (with summed momenta)**



# Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{df, 3 \rightarrow 3}} \frac{1}{iF_3} A_3$$


  
**row**   **matrices**                                   **column**  
**all in**  $\vec{k}, \ell, m$  **space**

---

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L, 2 \rightarrow 2}} iG \right] i\mathcal{M}_{L, 2 \rightarrow 2} iF$$

$$i\mathcal{M}_{L, 2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

# Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{df,3 \rightarrow 3} iF_3} A_3$$

↑ no poles      ↑ no poles      ↑ no poles

$C_L(E, \vec{P})$  diverges whenever  $iF_3 \frac{1}{1 - i\mathcal{K}_{df,3 \rightarrow 3} iF_3}$  diverges

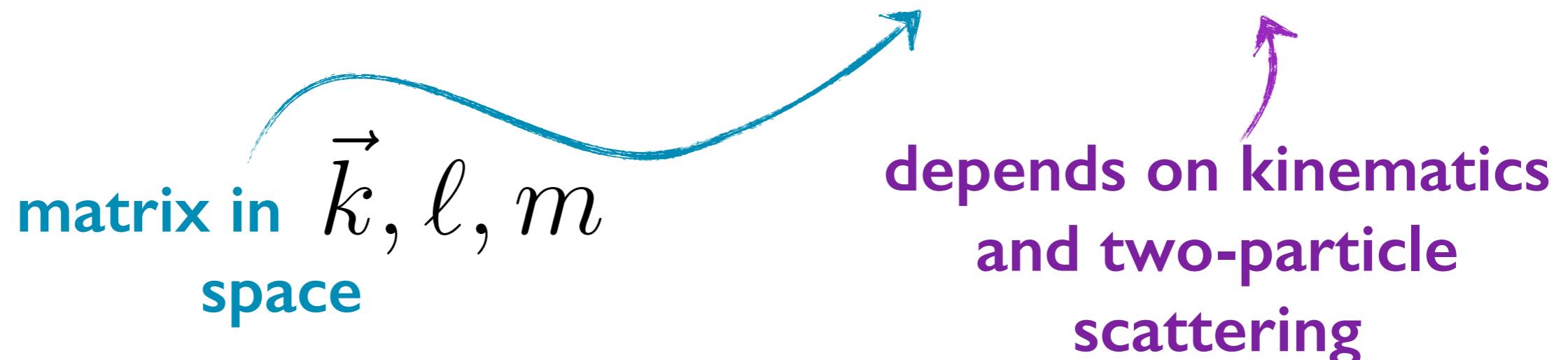
$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right]$$

$$i\mathcal{M}_{L,2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

# Three-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum  
is all solutions to

$$\Delta_{L,P}(E) = \det [1 - i\mathcal{K}_{\text{df},3 \rightarrow 3} iF_3] = 0$$



# Three-particle result

$$\Delta_{L,P}(E) = \det [1 - i\mathcal{K}_{\text{df},3 \rightarrow 3} iF_3] = 0$$

...is it useful?

**truncate in angular momentum  
to reduce to finite matrices**



**need relation between  $i\mathcal{M}_{3 \rightarrow 3}$  and  $i\mathcal{K}_{\text{df},3 \rightarrow 3}$**

**need to explore parametrizations of  $i\mathcal{K}_{\text{df},3 \rightarrow 3}$**

# Threshold expansion

**At weak coupling, perturbatively study  
finite-volume shift from threshold**

$$E = 3m + \mathcal{O}(1/L^3) \leftarrow \text{finite-volume shift}$$

**We find...**

$$E = 3m + \frac{12\pi a}{mL^3} \left[ 1 + A\frac{a}{L} + B\frac{a^2}{L^2} \right] + C_1 \frac{1}{L^6} - \frac{\mathcal{K}_{\text{df},3 \rightarrow 3,\text{thresh}}}{48m^3 L^6} + C_2 \frac{\log(mL)}{L^6}$$

$A, B, C_2$  agree unambiguously with earlier work

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507  
Tan, S. *Phys. Rev.* A78 (2008) 013636

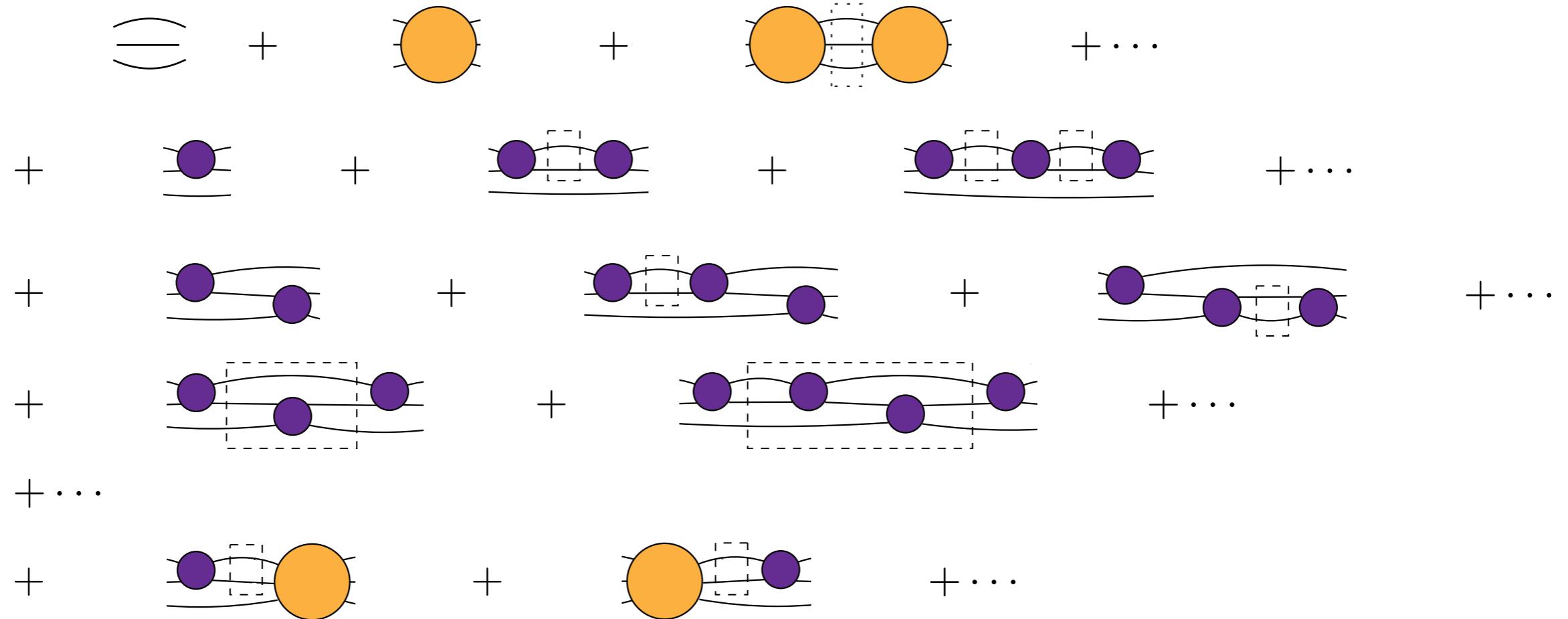
$C_1 - \frac{\mathcal{K}_{\text{df},3 \rightarrow 3,\text{thresh}}}{48m^3}$  related to non-relativistic contact interaction

# Relating $i\mathcal{K}_{\text{df}, 3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

$$C_L(E, \vec{P}) = \begin{array}{c} \text{Diagram 1: } \text{Two white circles connected by two horizontal lines. Dashed boxes group the left circle and the right circle.} \\ + \text{Diagram 2: } \text{A white circle connected to an orange circle, which is connected to another white circle. Dashed boxes group the first circle and the last circle.} \\ + \text{Diagram 3: } \text{Three white circles connected sequentially. Dashed boxes group the first circle and the last circle.} \\ + \dots \end{array}$$
$$+ \begin{array}{c} \text{Diagram 4: } \text{Two white circles connected by two horizontal lines. A purple circle is placed between them. Dashed boxes group the left circle and the right circle.} \\ + \text{Diagram 5: } \text{A white circle connected to a purple circle, which is connected to another white circle. Dashed boxes group the first circle and the last circle.} \\ + \text{Diagram 6: } \text{Three white circles connected sequentially. A purple circle is placed between the second and third circles. Dashed boxes group the first circle and the last circle.} \\ + \text{Diagram 7: } \text{Three white circles connected sequentially. A purple circle is placed between the first and second circles. Dashed boxes group the first circle and the last circle.} \\ + \dots \end{array}$$
$$+ \dots$$
$$+ \begin{array}{c} \text{Diagram 8: } \text{Two white circles connected by two horizontal lines. A purple circle is placed between them. An orange circle is placed to the right of the second circle. Dashed boxes group the first circle and the last circle.} \\ + \text{Diagram 9: } \text{A white circle connected to an orange circle, which is connected to another white circle. A purple circle is placed between the second and third circles. Dashed boxes group the first circle and the last circle.} \\ + \dots \end{array}$$

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L, 3 \rightarrow 3}$**

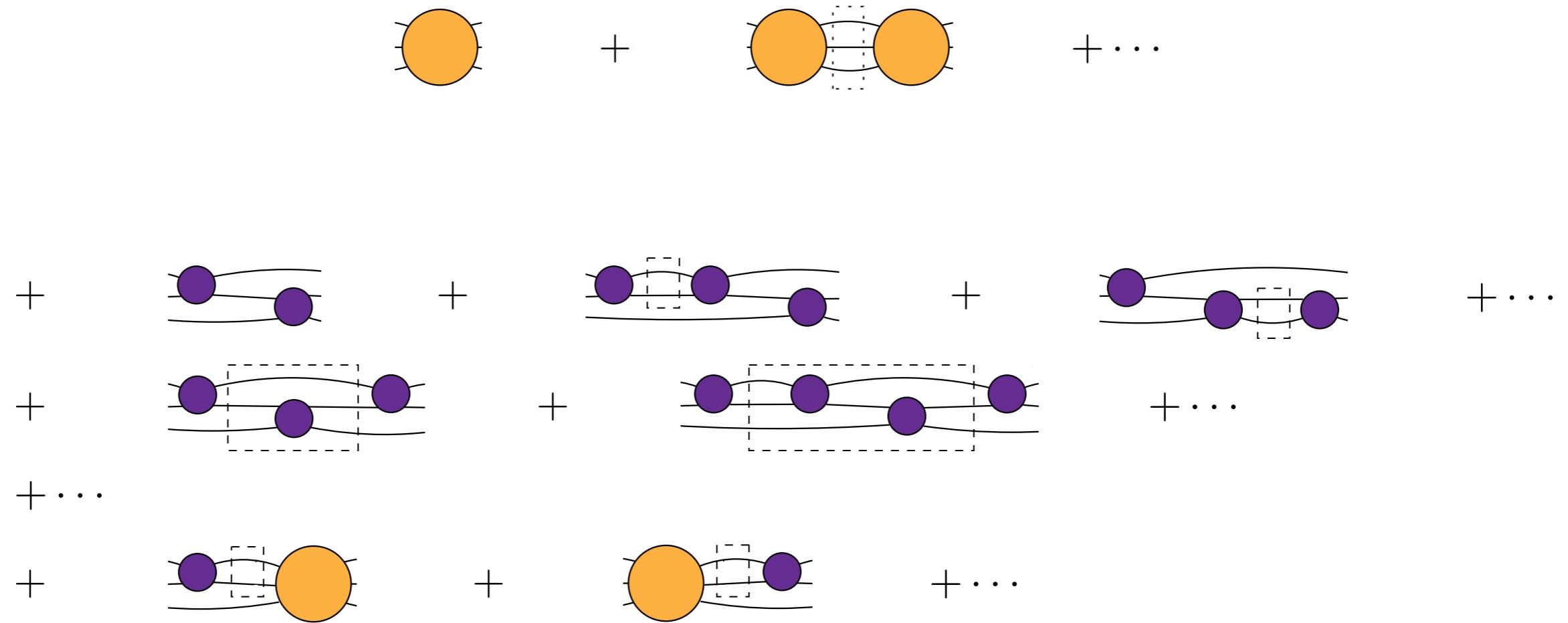
# Relating $i\mathcal{K}_{df,3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3 \rightarrow 3}$**

1. Amputate interpolating fields

# Relating $i\mathcal{K}_{df,3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$



**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3 \rightarrow 3}$**

**2. Drop disconnected diagrams**

# Relating $i\mathcal{K}_{\text{df},3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

$$i\mathcal{M}_{L,3 \rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 1: Orange circle with 3 external lines} \\ + \quad \text{Diagram 2: Two orange circles connected by a horizontal line, each with 3 external lines} \\ + \dots \\ \\ + \quad \text{Diagram 3: Two purple circles connected by a horizontal line, each with 3 external lines} \\ + \quad \text{Diagram 4: Three purple circles connected sequentially by two horizontal lines, each with 3 external lines} \\ + \quad \text{Diagram 5: Three purple circles connected sequentially by three horizontal lines, each with 3 external lines} \\ + \dots \\ \\ + \quad \text{Diagram 6: One orange circle and one purple circle connected by a horizontal line, each with 3 external lines} \\ + \quad \text{Diagram 7: One orange circle and one purple circle connected by a horizontal line, each with 3 external lines} \\ + \dots \end{array} \right\}$$

The equation shows a sum of Feynman-like diagrams representing the operator  $i\mathcal{M}_{L,3 \rightarrow 3}$ . The diagrams are categorized by color: orange and purple. Each diagram consists of a central node (circle) with three external lines, connected to other nodes or lines via internal lines. Dashed lines indicate symmetry or specific configurations.

**First we modify  $C_L(E, \vec{P})$  to define  $i\mathcal{M}_{L,3 \rightarrow 3}$**

**3. Symmetrize**

# Relating $i\mathcal{K}_{\text{df},3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

$$i\mathcal{M}_{L,3 \rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \\ + \dots \\ + \\ \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \dots \end{array} \right. \quad \begin{array}{c} \text{Diagram 6} \\ + \dots \end{array}$$

The equation shows the definition of  $i\mathcal{M}_{L,3 \rightarrow 3}$  as a sum of Feynman diagrams. The diagrams consist of three external lines (one solid black line and two dashed grey lines) and internal lines connecting them. Some internal lines are solid black, while others are dashed grey. The diagrams are organized into rows separated by plus signs. The first row contains Diagram 1 (a single yellow circle), Diagram 2 (two yellow circles connected by a horizontal line), and a plus sign followed by ellipses. The second row contains Diagram 3 (two purple circles connected by a horizontal line) and a plus sign. The third row contains Diagram 4 (three purple circles connected by a horizontal line) and a plus sign. The fourth row contains a plus sign and ellipses. The fifth row contains Diagram 5 (one purple circle connected to one yellow circle by a horizontal line) and a plus sign. The sixth row contains a plus sign followed by ellipses in a brace.

Replacing all loop momentum sums with  **$i$ -epsilonon prescription integrals** would give physical three-to-three scattering amplitude

# Relating $i\mathcal{K}_{\text{df},3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

**We find a simple form for  $i\mathcal{M}_{L,3 \rightarrow 3}$**

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[ \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iG \ i\mathcal{M}_{L,2 \rightarrow 2} [2\omega L^3] \right]$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iF \right]$$

$$\equiv \frac{iF}{2\omega L^3} \mathcal{L}_L \equiv \mathcal{R}_L \frac{iF}{2\omega L^3}$$

# Relating $i\mathcal{K}_{\text{df},3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

**We find a simple form for  $i\mathcal{M}_{L,3 \rightarrow 3}$**

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \mathcal{R}_L \right]$$

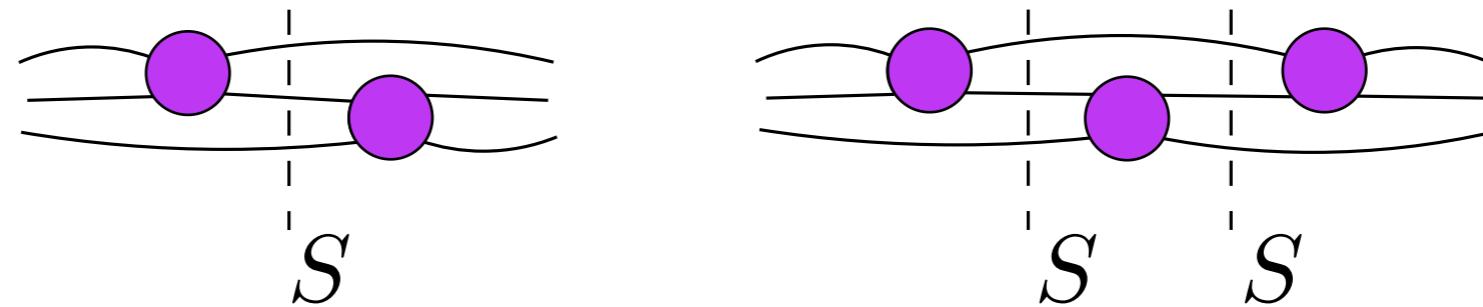
**Complete analysis with infinite volume limit**

$$i\mathcal{M}_{3 \rightarrow 3} = \lim_{L \rightarrow \infty} \Big|_{i\epsilon} i\mathcal{M}_{L,3 \rightarrow 3}$$

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \mathcal{R}_L \right]$$

**Recall**  $i\mathcal{M}_{\text{df},3 \rightarrow 3}$

$$\equiv i\mathcal{M}_{3 \rightarrow 3} - \left[ i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} + \int i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} S i\mathcal{M}_{2 \rightarrow 2} + \dots \right]$$



**It reappears here...**  $i\mathcal{M}_{df,3 \rightarrow 3} \equiv \lim_{L \rightarrow \infty} \left| \begin{array}{c} [i\mathcal{M}_{L,3 \rightarrow 3} - i\mathcal{D}_L] \\ i\epsilon \end{array} \right.$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[ \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iG \ i\mathcal{M}_{L,2 \rightarrow 2} [2\omega L^3] \right]$$

**encodes switches**

# Relating $i\mathcal{K}_{\text{df},3 \rightarrow 3}$ to $i\mathcal{M}_{3 \rightarrow 3}$

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[ \mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \mathcal{R}_L \right]$$
$$i\mathcal{M}_{3 \rightarrow 3} = \lim_{L \rightarrow \infty} \Bigg|_{i\epsilon} i\mathcal{M}_{L,3 \rightarrow 3}$$

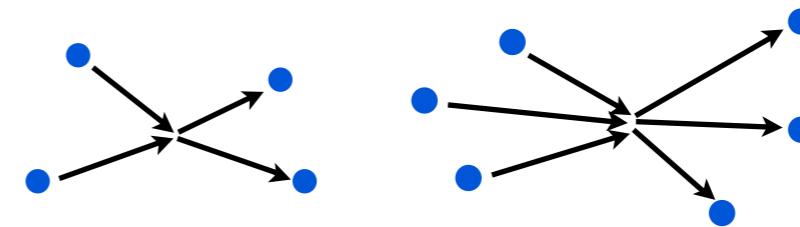
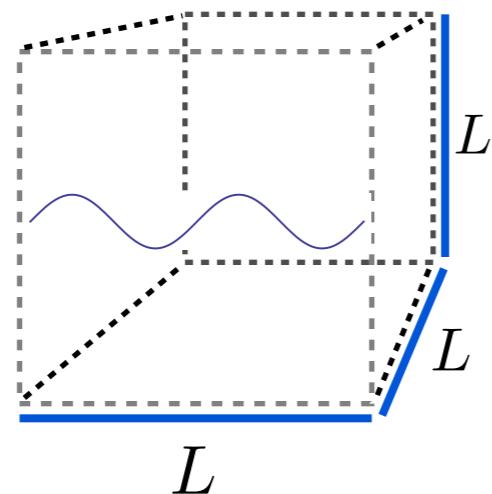
**Gives integral equation relating  $i\mathcal{K}_{\text{df},3 \rightarrow 3}$  to  $i\mathcal{M}_{3 \rightarrow 3}$**

**Completes formal story (for the setup considered!)**

**Relation only depends on on-shell scattering quantities**

# Summary

**Presented work relating  
finite-volume spectrum and three-to-three scattering.**



**Necessary first step for extracting any decay or scattering amplitude with more than two hadrons from Lattice QCD.**

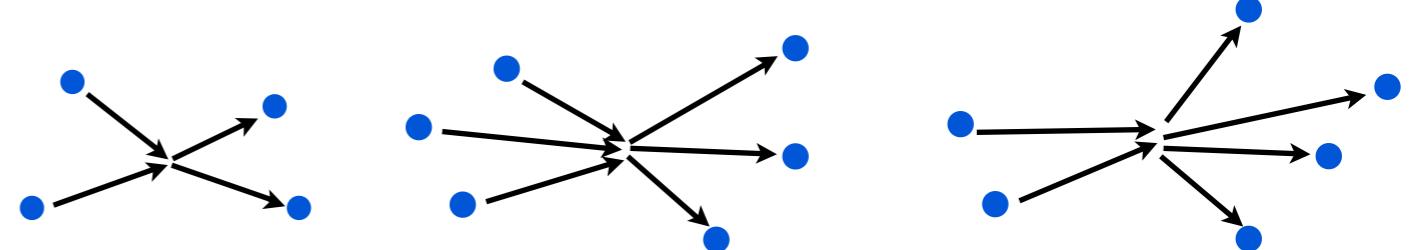
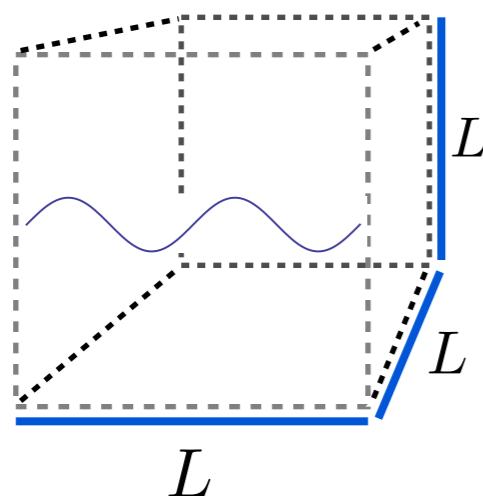
# Future work and Applications

**Generalize Lellouch-Lüscher method, to extract  
three-particle weak decays**

$$K \longrightarrow \pi\pi\pi$$

**Include non-identical, non-degenerate and spin-half particles**

**Extend mapping to four-particle states**



# Isotropic approximation

Following two particle case, suppose  $\mathcal{K}_{\text{df},3 \rightarrow 3}$  can be approximated to be isotropic (only depends on  $E^*$ )

$$\mathcal{K}_{\text{df},3 \rightarrow 3}(E_n^*) = -[F_{3,\text{iso}}(E_n, \vec{P}, L)]^{-1}$$

$$F_{3,\text{iso}} \equiv \sum_{\vec{k}, \vec{p}} F_{3;k,p}$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right]$$

# Three-particle result

At fixed  $(L, \vec{P})$  the finite-volume spectrum  $E_1, E_2, \dots$  is the set of solutions to

$$\Delta_{L,P}(E) = \det[1 - i\tilde{\mathcal{K}}_{df,3 \rightarrow 3} iF_3] = 0$$

where

$$iF_3 \equiv \frac{1}{2\omega L^3} \left[ -(2/3)iF_{\widetilde{\text{PV}}} + \frac{1}{[iF_{\widetilde{\text{PV}}}]^{-1} - [1 - i\tilde{\mathcal{K}}_{2 \rightarrow 2} iG]^{-1} i\tilde{\mathcal{K}}_{2 \rightarrow 2}} \right]$$

$$iF_{\widetilde{\text{PV}};k,k'} \equiv \delta_{k,k'} \frac{1}{2} \left[ \frac{1}{L^3} \sum_{\vec{a}} -\widetilde{\text{PV}} \int_{\vec{a}} \right] \frac{iQ(\vec{a}^*) Q^*(\vec{a}^*)}{2\omega_a 2\omega_{P-k-a} (E - \omega_k - \omega_a - \omega_{P-k-a})}$$

$$iG_{k,p} \equiv \frac{1}{2\omega_p L^3} \frac{iQ(\vec{p}^*) Q^*(\vec{k}^*)}{2\omega_{P-p-k} (E - \omega_p - \omega_k - \omega_{P-p-k})}$$

$$Q_{\ell,m}(\vec{k}^*) \equiv \sqrt{4\pi} Y_{\ell,m}(\hat{\vec{k}}^*) (k^*/q^*)^\ell$$

$$\text{with } \omega_k^2 = \vec{k}^2 + m^2 \text{ and } q^{*2} = (1/4)[E^2 - \vec{P}^2] - m^2$$

# Additional Material Concerning Differences Between Two- and Three- Particle Quantization

# New skeleton expansion

$$C_L(E, \vec{P}) = \text{(Diagram 1)} + \text{(Diagram 2)} + \text{(Diagram 3)} + \dots$$
$$+ \text{(Diagram 4)} + \text{(Diagram 5)} + \text{(Diagram 6)} + \dots$$
$$+ \text{(Diagram 7)} + \text{(Diagram 8)} + \text{(Diagram 9)} + \dots$$
$$+ \text{(Diagram 10)} + \text{(Diagram 11)} + \text{(Diagram 12)} + \dots$$
$$+ \dots$$
$$+ \text{(Diagram 13)} + \text{(Diagram 14)} + \text{(Diagram 15)} + \dots$$

Here I will only give first parts of derivation.

This is to illustrate certain points, needed to understand the final result.

## First Part: Sum “no-switch” diagrams

$$C_L^{(1)} \equiv \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots$$

call the bottom momentum  $k$

important finite-volume corrections only arise from  $k^0 = \omega_k$

$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots \right\}$$

**bottom propagator replaced with  $1/(2\omega_k)$**

# First Part: Sum “no-switch” diagrams

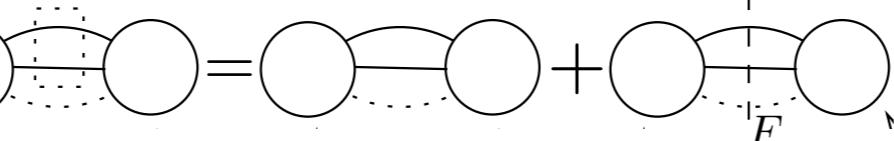
$$C_L^{(1)} \equiv \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots$$

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**bottom propagator replaced with  $1/(2\omega_k)$**

Next substitute  and regroup by F-cuts

$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \left\{ \text{Diagram} + \text{Diagram} \right\}$$

$$+ \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots \right\}$$

$$= \text{Diagram} + \text{Diagram}$$

$$= \langle \text{Diagram} \rangle + \langle \text{Diagram} \rangle + \langle \text{Diagram} \rangle + \dots$$

$$= i\mathcal{M}_{2 \rightarrow 2}$$

# First Part: Sum “no-switch” diagrams

$$C_L^{(1)} \equiv \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots$$

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$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots \right\}$$

Next substitute and regroup by F-cuts

$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \left\{ \text{Diagram} + \text{Diagram} \right\} \quad \text{Can this be replaced with an integral? No!}$$

$$+ \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \dots \right\}$$

$$= \text{Diagram} + \text{Diagram}$$

$$= \langle \text{Diagram} \rangle + \langle \text{Diagram} \rangle + \langle \text{Diagram} \rangle + \dots$$

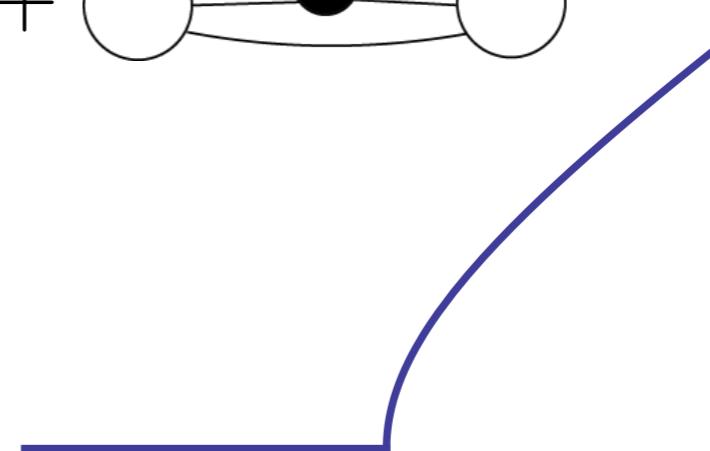
$$= i\mathcal{M}_{2 \rightarrow 2}$$

# Main Lesson Number 1: Cusp effects

$$\frac{1}{L^3} \sum_{\vec{k}} f(\vec{k}) = \int_{\vec{k}} f(\vec{k}) + \mathcal{O}(e^{-mL})$$

only holds if  $f(\vec{k})$  is an infinitely differentiable function with width  $\sim m$ .

$$C_{\infty}^{(1)}(\vec{k}) \equiv \text{Diagram: two circles connected by two horizontal lines} + \text{Diagram: three circles connected by two horizontal lines, one circle is black}$$



has a unitary cusp at threshold

$$(E - \omega_k)^2 - (\vec{P} - \vec{k})^2 = 4m^2$$

To remove cusp

$i\epsilon$  prescription

principal value

$\tilde{\text{PV}}$

(analytically continue below threshold, then interpolate to standard subthreshold form)

Polejaeva, K. and Rusetsky, A. *Eur. Phys. J.* A48 (2012) 67

$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \left\{ \text{Diagram: two circles connected by two horizontal lines} + \text{Diagram: one circle with a black dot inside} \right\}$$

Can this be replaced with an integral? No!

$$+ \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram: alternating gray and black circles along a horizontal line} + \dots \right\}$$

$$= \langle \text{Diagram: two purple circles connected by two horizontal lines} \rangle + \langle \text{Diagram: three purple circles connected by three horizontal lines} \rangle + \langle \text{Diagram: four purple circles connected by four horizontal lines} \rangle + \dots$$

$$= i\mathcal{M}_{2 \rightarrow 2}$$

$$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \left\{ \text{Diagram with one purple circle} + \text{Diagram with one blue circle} \right\}$$

**But if we change pole prescription, then we can make the replacement**

$$+ \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k} \left\{ \text{Diagram with one blue circle} + \text{Diagram with one purple circle} + \dots \right\}$$

$F_{\widetilde{\text{PV}}}$

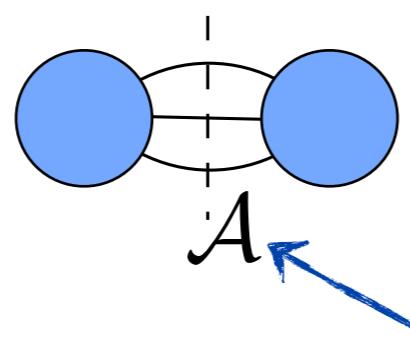
$$= \langle \text{Diagram with two purple circles} \rangle + \langle \text{Diagram with three purple circles} \rangle + \langle \text{Diagram with four purple circles} \rangle + \dots$$

$i\tilde{\mathcal{K}}_{2 \rightarrow 2}$

$\text{PV}$

**All three-particle poles now have principal-value prescription**

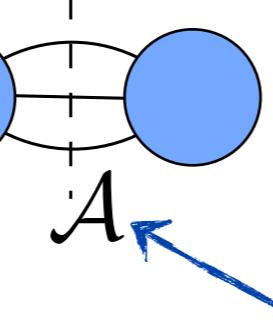
Deduce

$$C_L^{(1)} = \tilde{C}_{\infty}^{(1)} + \text{Diagram}$$


$$[\mathcal{A}] \equiv \frac{iF_{\widetilde{\text{PV}}}}{2\omega L^3} \frac{1}{1 + \tilde{\mathcal{K}}_{2 \rightarrow 2} F_{\widetilde{\text{PV}}}}$$

**think of this as a new cut,  
like F it puts neighbors on-shell**

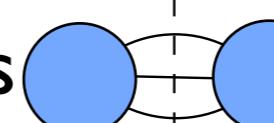
Deduce

$$C_L^{(1)} = \tilde{C}_\infty^{(1)} + \text{Diagram}$$


$$[\mathcal{A}] \equiv \frac{iF_{\widetilde{\text{PV}}}}{2\omega L^3} \frac{1}{1 + \tilde{\mathcal{K}}_{2 \rightarrow 2} F_{\widetilde{\text{PV}}}}$$

**think of this as a new cut,  
like  $\mathbf{F}$  it puts neighbors on-shell**

## Main Lesson Number 2: Matrix structure

Finite volume residue terms (such as ) are of the form:

(row vector)  $\times$  (matrix)  $\times$  (column vector), acting on product space

[finite-volume momentum]  $\times$  [angular momentum]

For example,  $[\mathcal{A}]$  is built from

$$F_{\widetilde{\text{PV}};k',\ell',m';k,\ell,m} \equiv \delta_{k',k} F_{\widetilde{\text{PV}};\ell',m';\ell,m}(E - \omega_k, \vec{P} - \vec{k}) \quad \vec{k} = \vec{k}' \in (2\pi/L)\mathbb{Z}^3$$

$$\tilde{\mathcal{K}}_{2 \rightarrow 2;k',\ell',m';k,\ell,m} \equiv \delta_{k',k} \tilde{\mathcal{K}}_{2 \rightarrow 2;\ell',m';\ell,m}(E - \omega_k, \vec{P} - \vec{k})$$

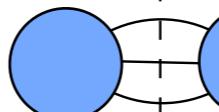
Deduce

$$C_L^{(1)} = \tilde{C}_\infty^{(1)} + \text{Diagram}$$

$$[\mathcal{A}] \equiv \frac{iF_{\widetilde{\text{PV}}}}{2\omega L^3} \frac{1}{1 + \tilde{\mathcal{K}}_{2 \rightarrow 2} F_{\widetilde{\text{PV}}}}$$

think of this as a new cut,  
like  $\mathbf{F}$  it puts neighbors on-shell

## Main Lesson Number 2: Matrix structure

Finite volume residue terms (such as ) are of the form:

(row vector)  $\times$  (matrix)  $\times$  (column vector), acting on product space

[finite-volume momentum]  $\times$  [angular momentum]

Observe that  $\vec{k}, \ell, m$  parametrizes three particles with fixed  $(E, \vec{P})$



## Second part: Sum “one-switch” diagrams

$$C_L^{(2)} = \dots + \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagram consists of three horizontal lines representing momenta. Purple circles represent vertices where two lines meet. A blue dashed oval encloses the first vertex from the left. Dashed boxes enclose the second and third vertices from the left.

In this case we have two “spectator-momenta”  
(momenta that do not appear in two-particle loops)

$$C_L^{(2)} = \tilde{C}_\infty^{(2)} + \text{Diagram} + \dots$$

The diagram shows a sequence of blue circles connected by horizontal lines. Between the first and second circles, and between the second and third circles, there are vertical dashed lines labeled  $\mathcal{A}$ . The  $\dots$  indicates further terms in the sum.

**stands for terms that  
modify endcaps of  $C_L^{(1)}$**

Between  $\mathcal{A}$  factors we have first contribution to three-to-three scattering

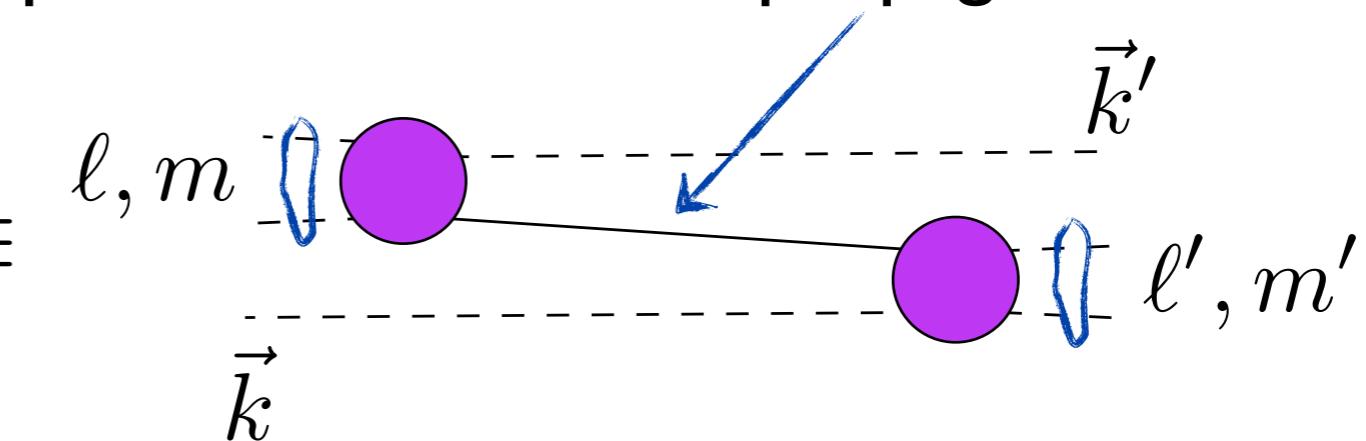
$$i\tilde{\mathcal{K}}_{3 \rightarrow 3; k', \ell', m'; k, \ell, m}^{(2, \text{unsym})} \equiv \ell, m \quad \begin{array}{c} \text{Diagram} \\ \text{with} \\ \vec{k} \text{ and} \\ \vec{k}' \text{ momenta} \\ \text{and} \\ \ell', m' \text{ indices} \end{array}$$

The diagram shows two purple circles connected by a horizontal line. Dashed lines extend from the top and bottom of each circle, labeled  $\vec{k}'$  and  $\vec{k}$  respectively. Indices  $\ell, m$  are associated with the left circle, and indices  $\ell', m'$  are associated with the right circle.

# Main Lesson Number 3: On-shell divergences

Certain external momenta put the intermediate propagator on-shell

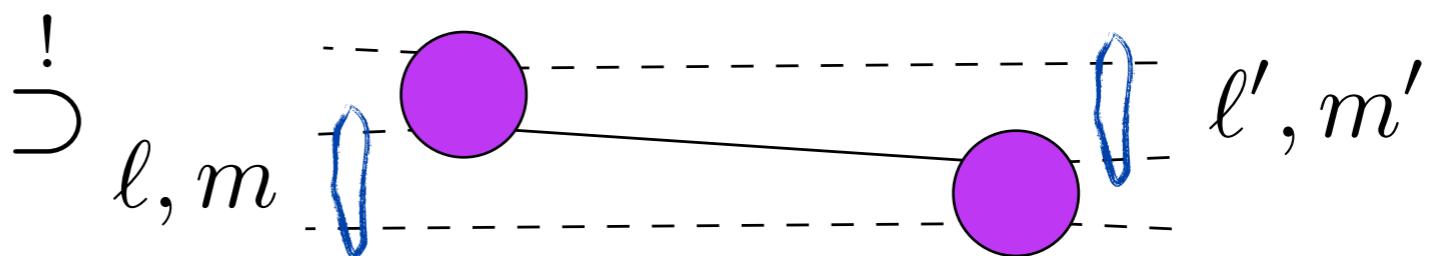
$$i\tilde{\mathcal{K}}_{3 \rightarrow 3; k', \ell', m'; k, \ell, m}^{(2, \text{unsym})}$$



This implies that this diagram, and indeed also the full  
 $i\tilde{\mathcal{K}}_{3 \rightarrow 3}$  **has physical singularities above threshold**  
**nothing to do with bound states**

This is a problem because K-matrix is symmetric in external momenta

$$i\tilde{\mathcal{K}}_{3 \rightarrow 3; k', \ell', m'; k, \ell, m}$$

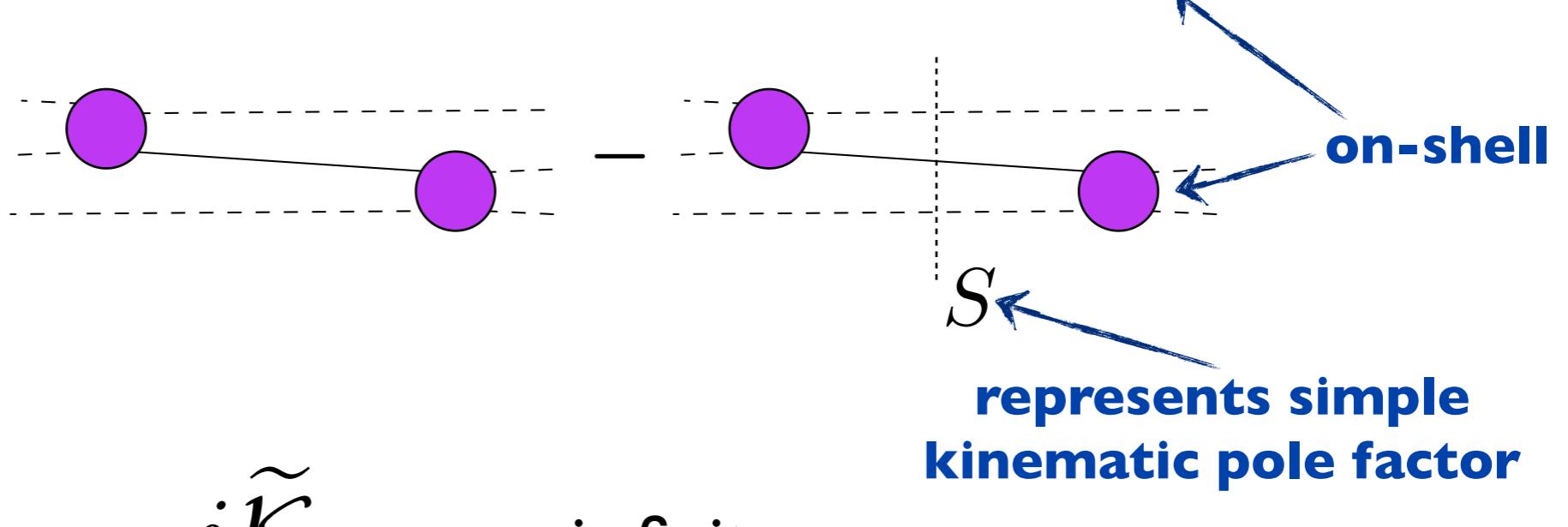


But this would demand decomposing a singular function in  $Y_{\ell, m}$

**The decomposition is not valid!**

## Resolution: Introduce

$$i\tilde{\mathcal{K}}_{df,3 \rightarrow 3}^{(2,\text{unsym})} \equiv i\tilde{\mathcal{K}}_{3 \rightarrow 3}^{(2,\text{unsym})} - i\tilde{\mathcal{K}}_{2 \rightarrow 2}S i\tilde{\mathcal{K}}_{2 \rightarrow 2}$$



$$i\tilde{\mathcal{K}}_{df,3 \rightarrow 3} \text{ is finite:}$$

Can decompose in harmonics and truncate expansion at low energies

The approach of separating out singularities like this was first suggested over 40 years ago (Rubin et al. *PR 146-6* (1966))

**It makes sense to recover singularity-free quantity from finite-volume spectrum.**

**Then add singular terms back in.**

This pattern of separating out singularities persists to all orders

Define

$$i\tilde{\mathcal{K}}_{df,3 \rightarrow 3} \equiv i\tilde{\mathcal{K}}_{3 \rightarrow 3} - \left[ i\tilde{\mathcal{K}}_{2 \rightarrow 2} S i\tilde{\mathcal{K}}_{2 \rightarrow 2} + \int i\tilde{\mathcal{K}}_{2 \rightarrow 2} S i\tilde{\mathcal{K}}_{2 \rightarrow 2} S i\tilde{\mathcal{K}}_{2 \rightarrow 2} + \dots \right]$$

only on-shell amplitudes here

infinite series built with factors of  $S i\tilde{\mathcal{K}}_{2 \rightarrow 2}$

Definition arises from analyzing all two-to-two diagrams

$$i\tilde{\mathcal{K}}_{df,3 \rightarrow 3; k', l', m'; k, l, m}$$

is the natural observable to extract from  
the finite-volume spectrum

# Review Lessons

1. Need modified principal value to remove cusp effects
2. In the three particle case, all matrices act on product space [finite-volume momentum]x[angular momentum]  
In other words, they have indices  $\vec{k}, \ell, m$

**needed to describe  
three particles**



3. Singularities in  $i\tilde{\mathcal{K}}_{3 \rightarrow 3}$  invalidate decomposition in  $Y_{\ell,m}$   
Resolution is to introduce  $i\tilde{\mathcal{K}}_{df,3 \rightarrow 3}$

$$i\tilde{\mathcal{K}}_{df,3 \rightarrow 3} \equiv i\tilde{\mathcal{K}}_{3 \rightarrow 3} - \left[ \text{diagram with two particles} + \text{diagram with three particles} + \dots \right]$$

The equation shows the definition of  $i\tilde{\mathcal{K}}_{df,3 \rightarrow 3}$  as the difference between  $i\tilde{\mathcal{K}}_{3 \rightarrow 3}$  and a sum of diagrams. The first diagram in the sum shows two purple circles connected by horizontal lines, with a vertical dashed line labeled  $S$  passing through them. The second diagram shows three purple circles connected by horizontal lines, with two vertical dashed lines labeled  $S$  passing through them.

This object arises naturally in finite-volume analysis.

# Intro- and Two-Particle Material

# What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

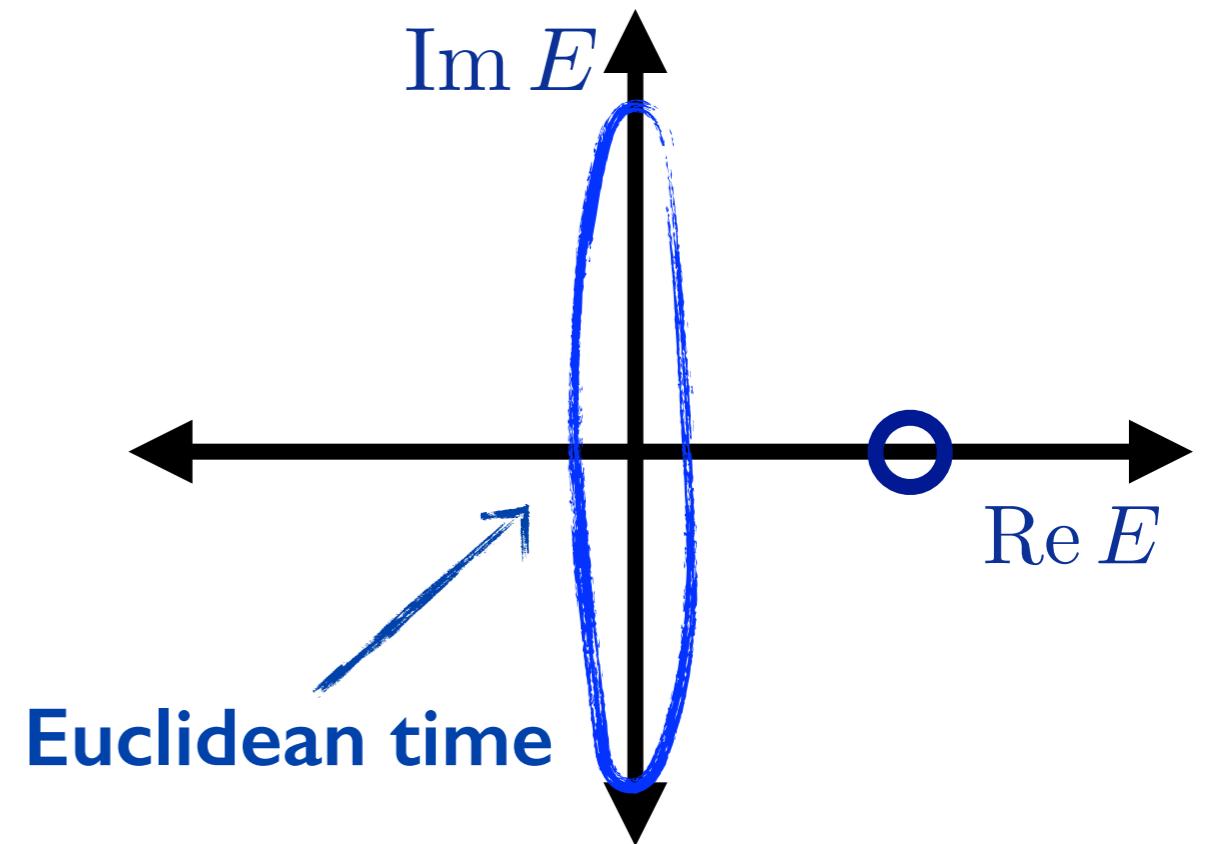
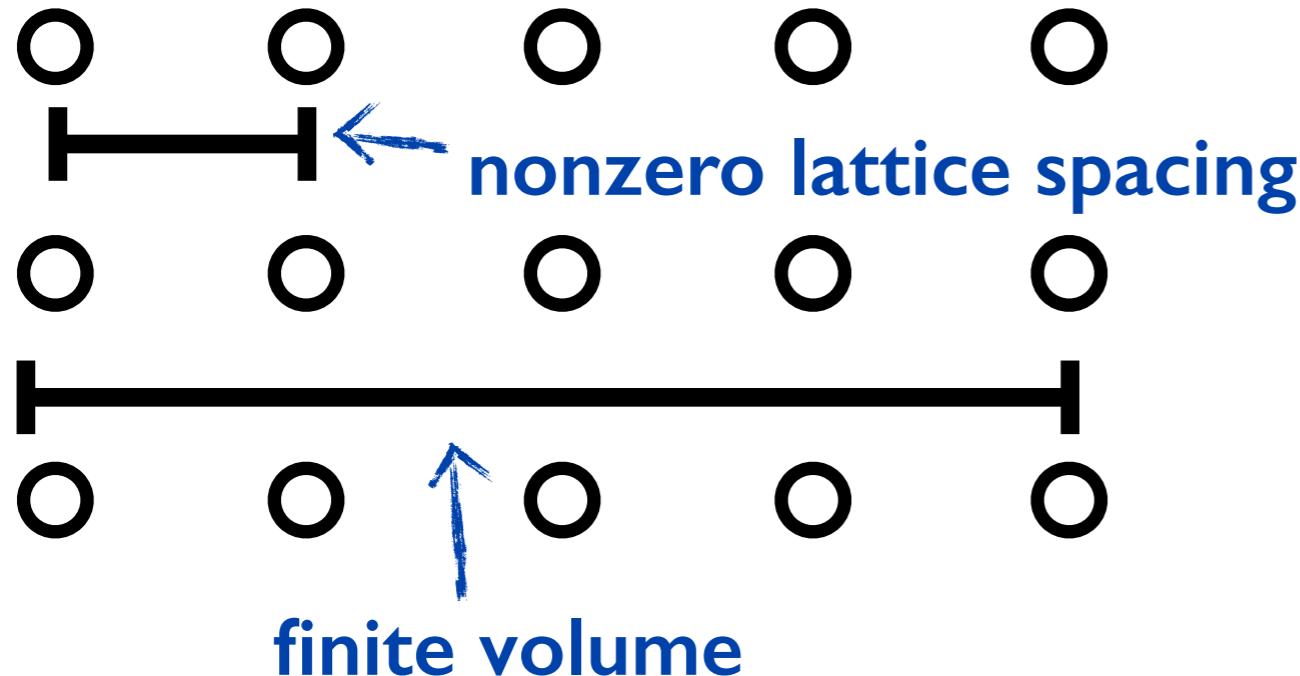
$$\langle T\phi_1 \cdots \phi_n \rangle = \int \mathcal{D}\phi e^{iS} \phi_1 \cdots \phi_n$$

# What can we extract from LQCD?

We are trying to evaluate a difficult integral numerically

$$\langle T\phi_1 \cdots \phi_n \rangle_{\text{Euc, latt, fv}} = \int \prod_i^N d\phi_i e^{-S} \phi_1 \cdots \phi_n$$

To do so we have to make three compromises



# What can we extract from LQCD?

**Not possible to directly calculate**

$$\langle \underline{\pi\pi} | \pi\pi \rangle$$

$$\langle \pi\pi\pi | \underline{\pi\pi\pi} \rangle$$

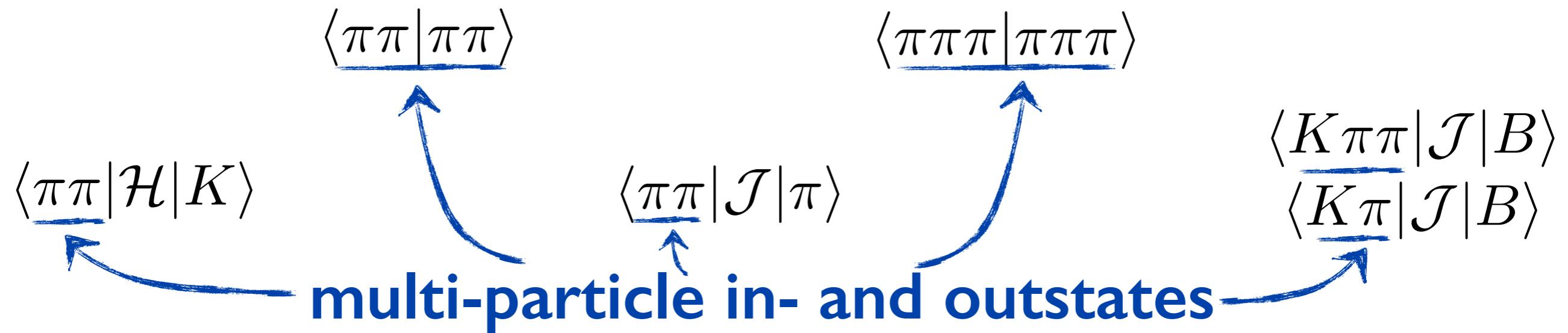
$$\langle \underline{\pi\pi} | \mathcal{H} | K \rangle$$

$$\langle \underline{\pi\pi} | \mathcal{J} | \pi \rangle$$

$$\begin{aligned} &\langle \underline{K\pi\pi} | \mathcal{J} | B \rangle \\ &\langle \underline{\overline{K}\pi} | \mathcal{J} | B \rangle \end{aligned}$$

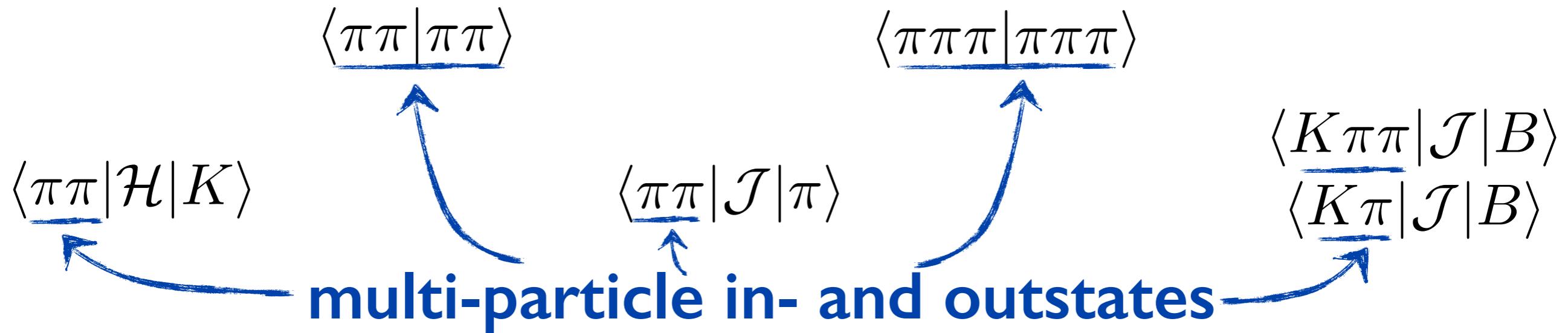
# What can we extract from LQCD?

**Not possible to directly calculate**



# What can we extract from LQCD?

**Not possible to directly calculate**



$$\langle \pi\pi | \pi\pi \rangle = \text{Amputate and put on-shell} \\ \langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle$$

$$\langle \pi\pi | \mathcal{H} | K \rangle = \text{Amputate and put on-shell} \\ \langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \mathcal{H}(x) \tilde{K}(P) | 0 \rangle$$

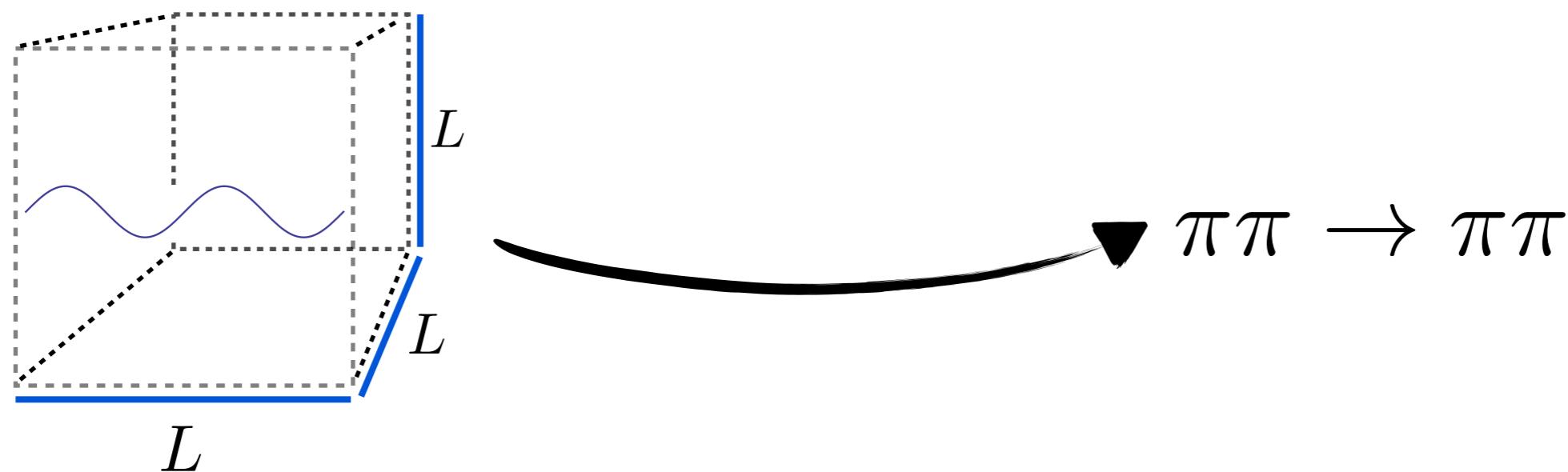
**Requires Minkowski momenta and infinite-volume**

In 1991 M. Lüscher found a method to circumvent this issue and extract  $\pi\pi \rightarrow \pi\pi$  scattering from LQCD.

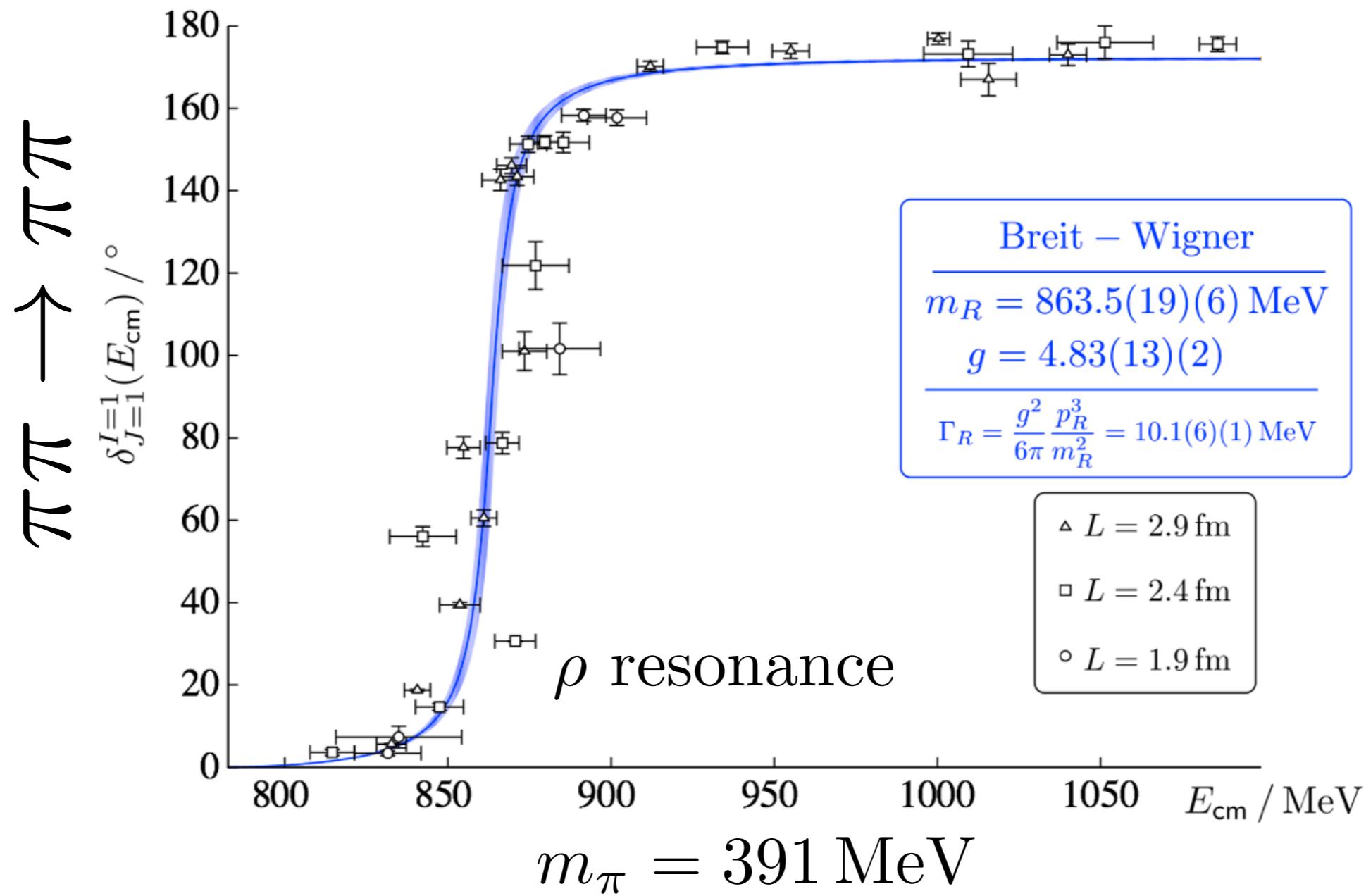
Lüscher, M. *Nucl. Phys.* B354, 531-578 (1991)

His key insight was to use **finite-volume** as a tool.

He gave a mapping between **finite-volume spectrum** and **elastic pion scattering amplitude**.



Lüscher's method has led to a large body of work extracting phase shifts from Lattice QCD.



from Dudek, Edwards, Thomas in *Phys.Rev. D87* (2013) 034505

However, there is **no general method** for extracting scattering amplitudes involving more than two hadrons.

As a result LQCD cannot investigate...

However, there is **no general method** for extracting scattering amplitudes involving more than two hadrons.

As a result LQCD cannot investigate...

**resonances which decay into more than two hadrons**

$$\omega(782) \rightarrow \pi\pi\pi$$

$$N(1440) \rightarrow N\pi\pi$$

**three-body forces**

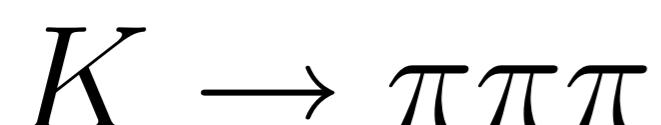
$$\pi\pi\pi \rightarrow \pi\pi\pi$$

$$NNN \rightarrow NNN$$

However, there is **no general method** for extracting scattering amplitudes involving more than two hadrons.

As a result LQCD cannot investigate...

**weak decays coupled to channels containing more than two hadrons**



*Need strong scattering to relate lattice weak matrix elements to physical decay amplitudes*

Lellouch, L. & Lüscher, M.  
*Commun. Math. Phys.* 219, 31-44 (2001)

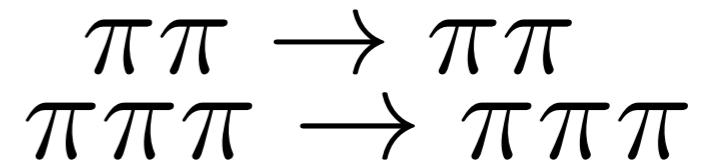
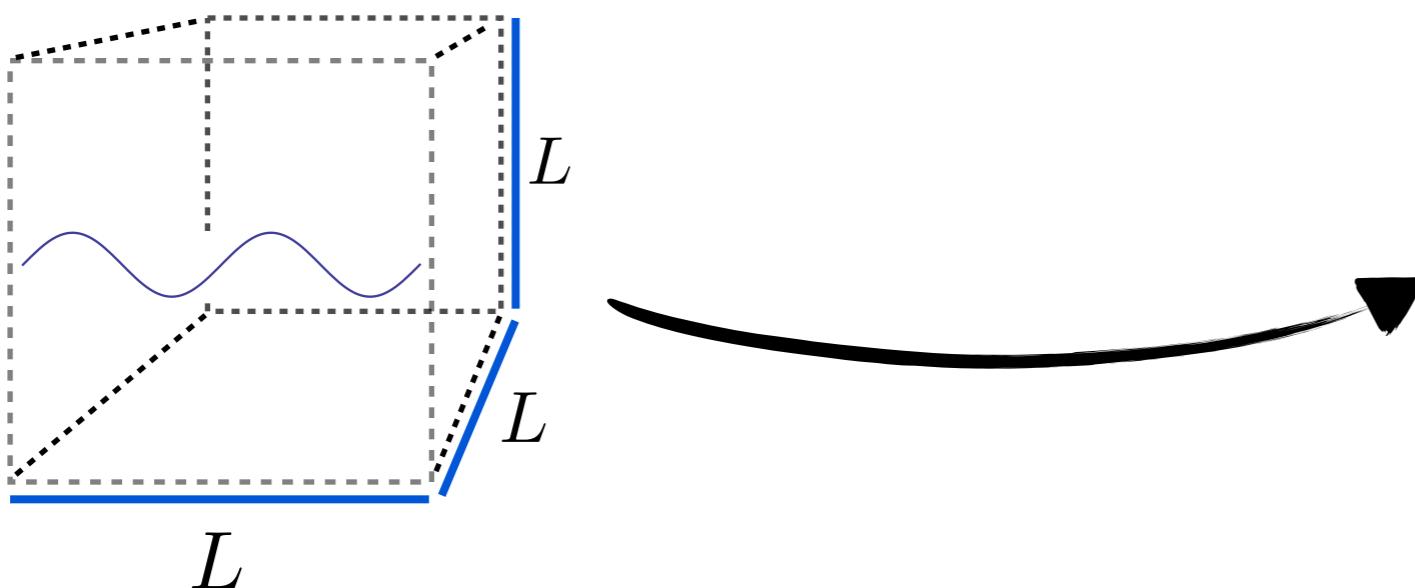
In recent years important progress has been made towards extracting three-particle scattering.

Polejaeva, K. and Rusetsky, A. *Eur. Phys. J.* A48 (2012) 67

Briceno, R. A. and Davoudi, Z. *Phys. Rev.* D87 (2013) 094507

However a relativistic, model-independent method is still unknown.

This is the focus of today's talk.



# Outline

Detailed set-up

Two particles in a box

Three particles in a box

Conclusion

# Particle content

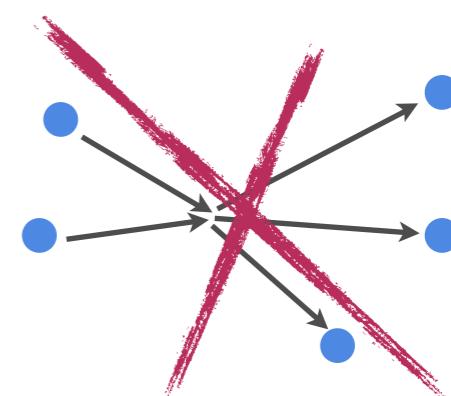
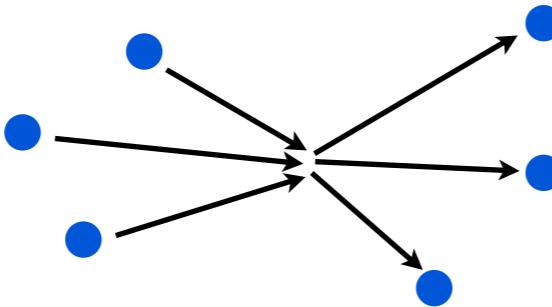
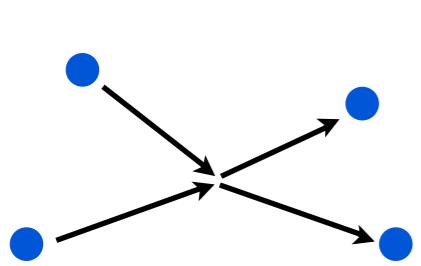
Single scalar, mass  $m$



all results for  
identical scalars

Relativistic field theory

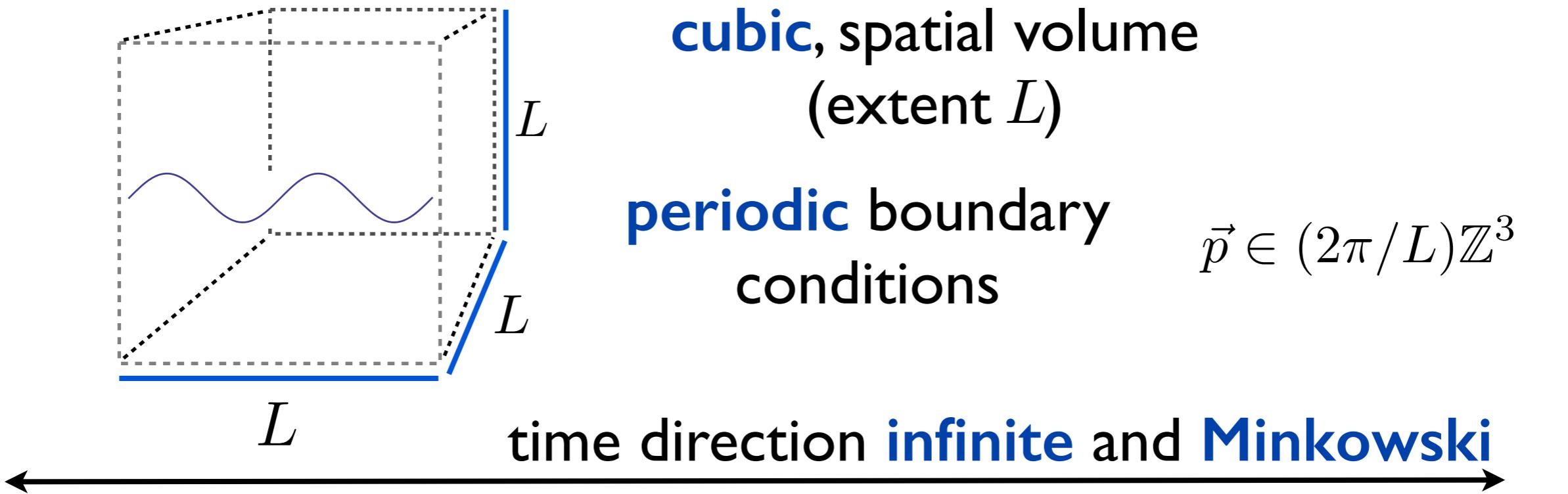
$\mathbb{Z}_2$  symmetry



(For pions in QCD this is G-parity)

**Include all vertices  
with even number of legs**

# Finite volume



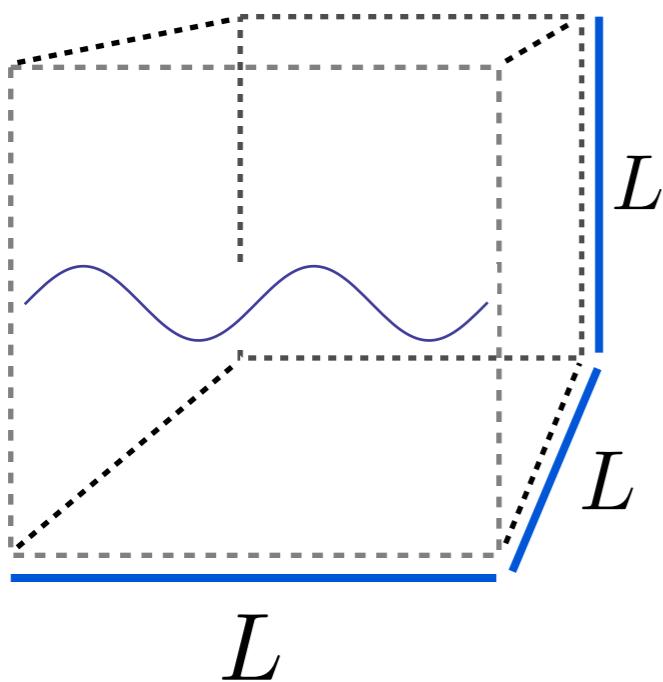
Take  $L$  large enough to ignore  $e^{-mL}$

dropped throughout!

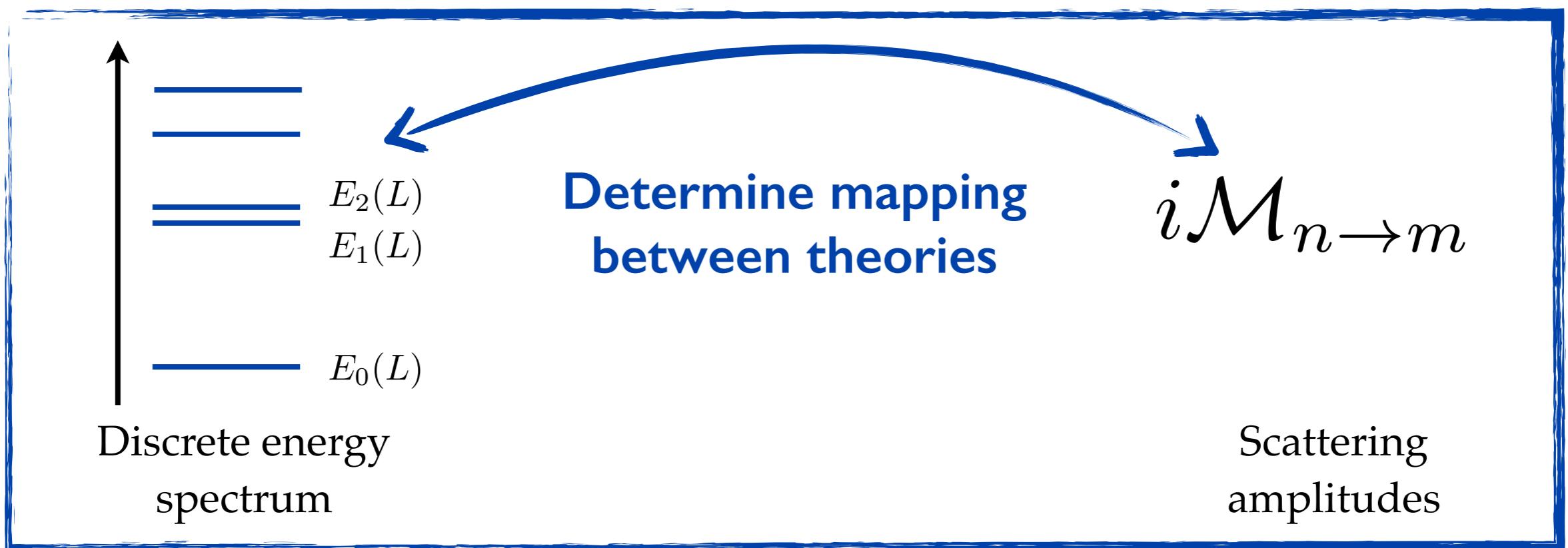
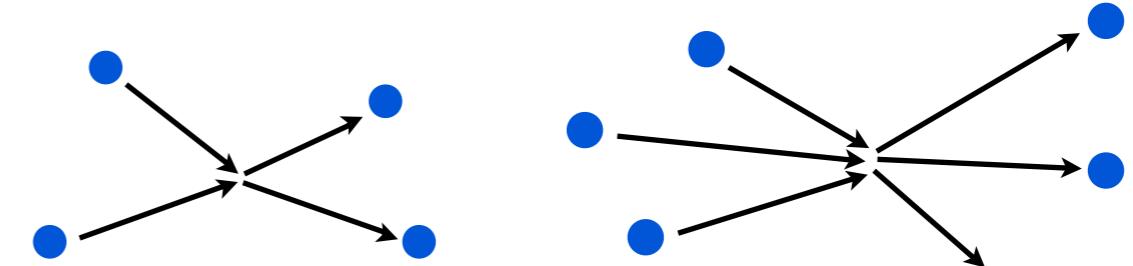
Take space to be continuous

lattice spacing set  
to zero

# Finite volume



# Infinite volume



## Determine relation using finite-volume correlator

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

## Determine relation using finite-volume correlator

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$

interpolating field

## Determine relation using finite-volume correlator

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

energy  $E$ , momentum  $\vec{P} = (2\pi/L)\vec{n}_P$

CM energy  $E^{*2} \equiv E^2 - \vec{P}^2$



interpolating field

**At fixed  $L, \vec{P}$ , poles in  $C_L$  give finite-volume spectrum**

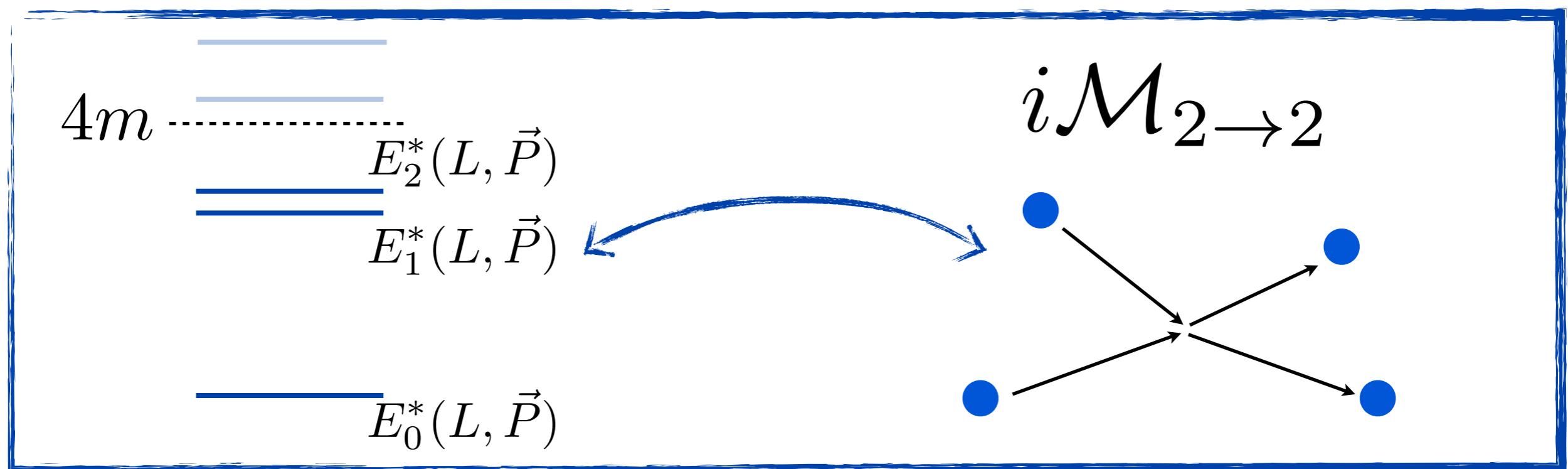
We calculate  $C_L(E, \vec{P})$  to all orders in perturbation theory and determine condition of divergence.

# First, two particles in a box

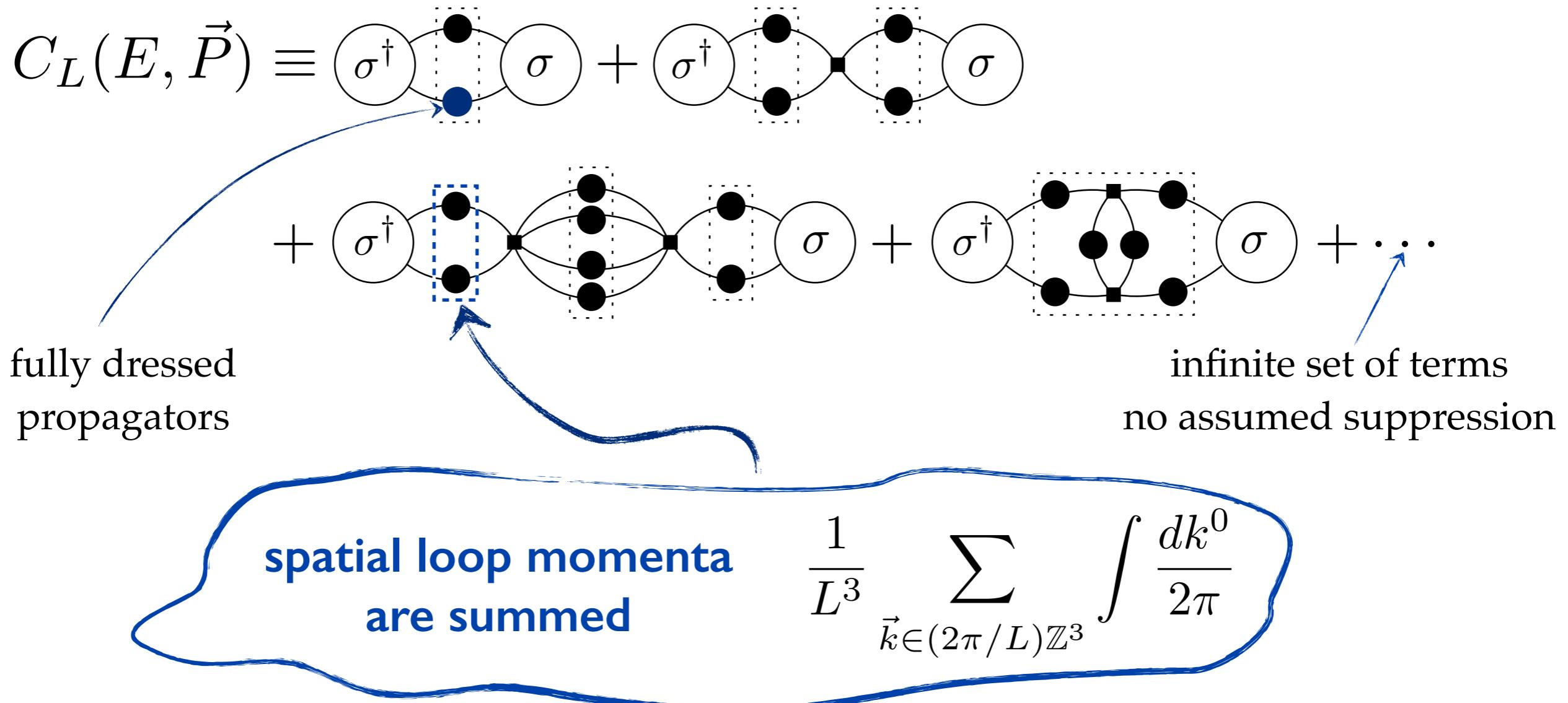
$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{i(Ex^0 - \vec{P} \cdot \vec{x})} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

Require  $E^* < 4m$

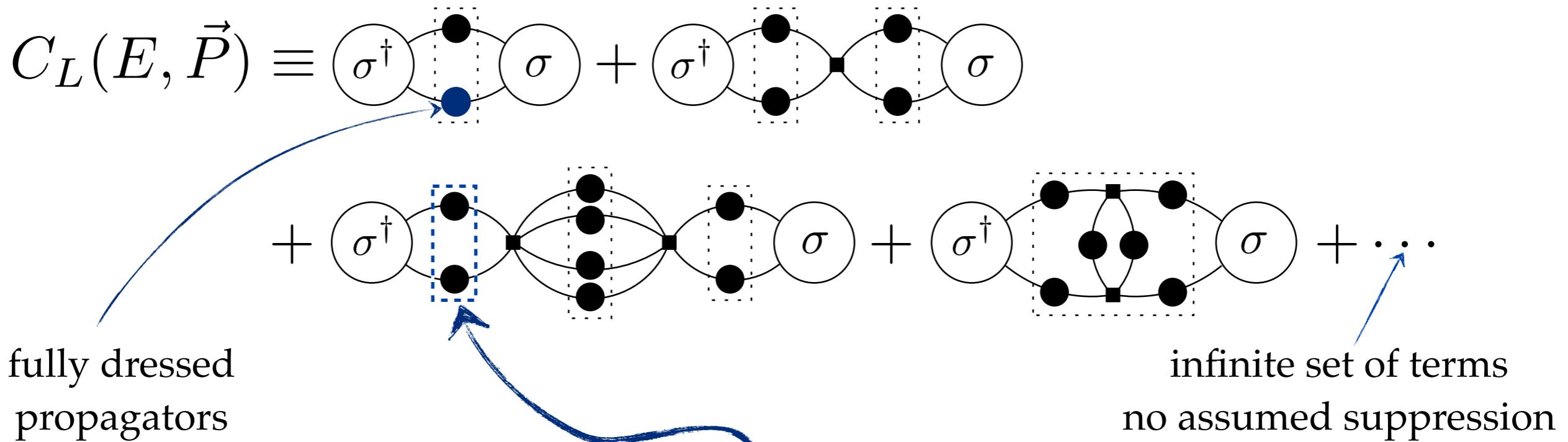
**even-particle quantum numbers**



Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys. B* 727, 218-243  
(2005)



$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



**spatial loop momenta  
are summed**

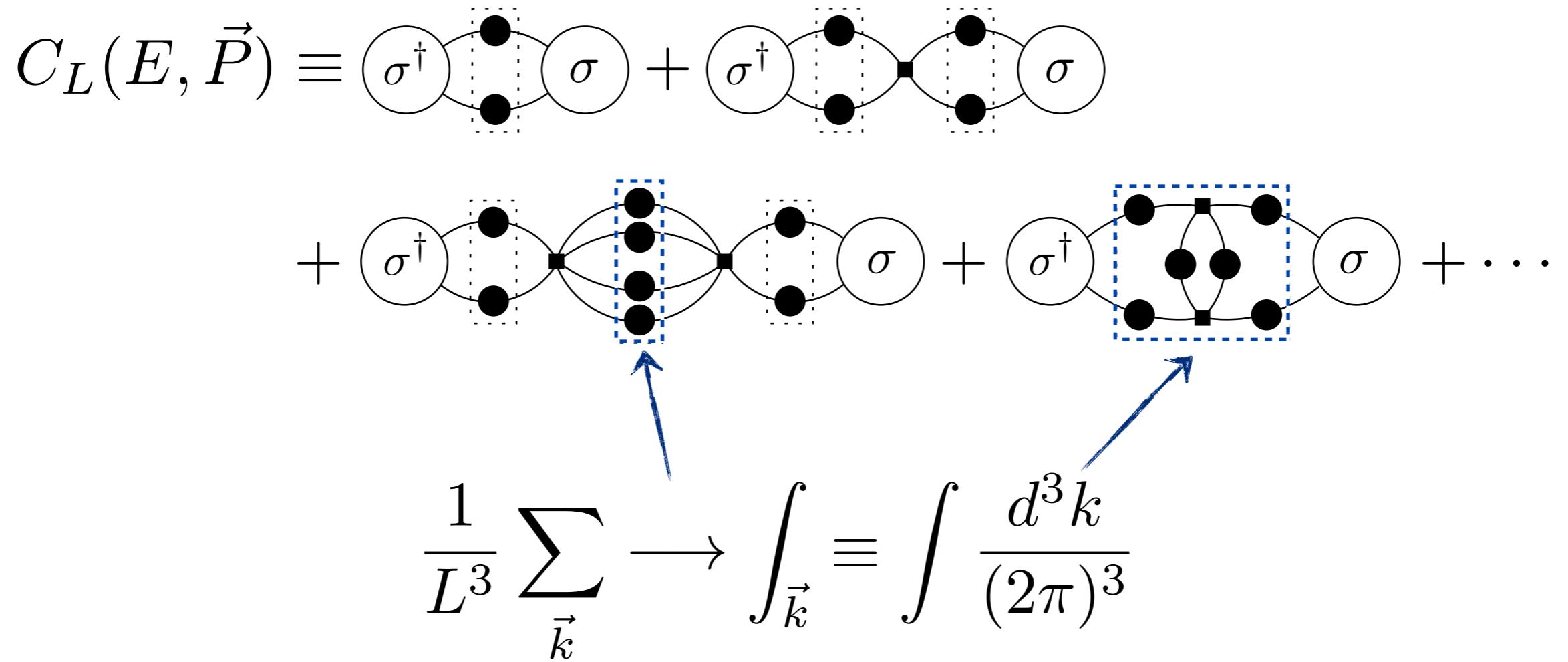
$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

## Key observation:

If particles in summed loops cannot all go on shell, then replace

$$\frac{1}{L^3} \sum_{\vec{k}} \rightarrow \int \frac{d^3 k}{(2\pi)^3}$$

**difference is order**  
 $e^{-mL}$



Since  $E^* < 4m$ , **only two particles with total momentum  $(E, \vec{P})$  can go on-shell**

$$C_L(E, \vec{P}) = \sigma^\dagger \text{---} \sigma + \sigma^\dagger \text{---} \sigma$$

these loops are now integrated

$$+ \sigma^\dagger \text{---} \sigma + \sigma^\dagger \text{---} \sigma + \dots$$

$$C_L(E, \vec{P}) = \text{Diagram} + \text{Diagram}$$

these loops are now integrated

$$+ \text{Diagram} + \text{Diagram} + \dots$$

$$C_L(E, \vec{P}) = \text{Diagram}$$

infinite-volume  
Bethe-Salpeter kernel

$$+ \text{Diagram} \left\{ \text{Diagram} + \text{Diagram} + \dots \right\} + \text{Diagram} + \dots$$

$$C_L(E, \vec{P}) = \text{Diagram} + \text{Diagram}$$

$$+ \text{Diagram} + \text{Diagram} + \dots$$

$$C_L(E, \vec{P}) = \sigma^\dagger \circlearrowleft \sigma + \sigma^\dagger \circlearrowleft iK \circlearrowright \sigma + \sigma^\dagger \circlearrowleft iK \circlearrowright iK \circlearrowright \sigma + \dots$$

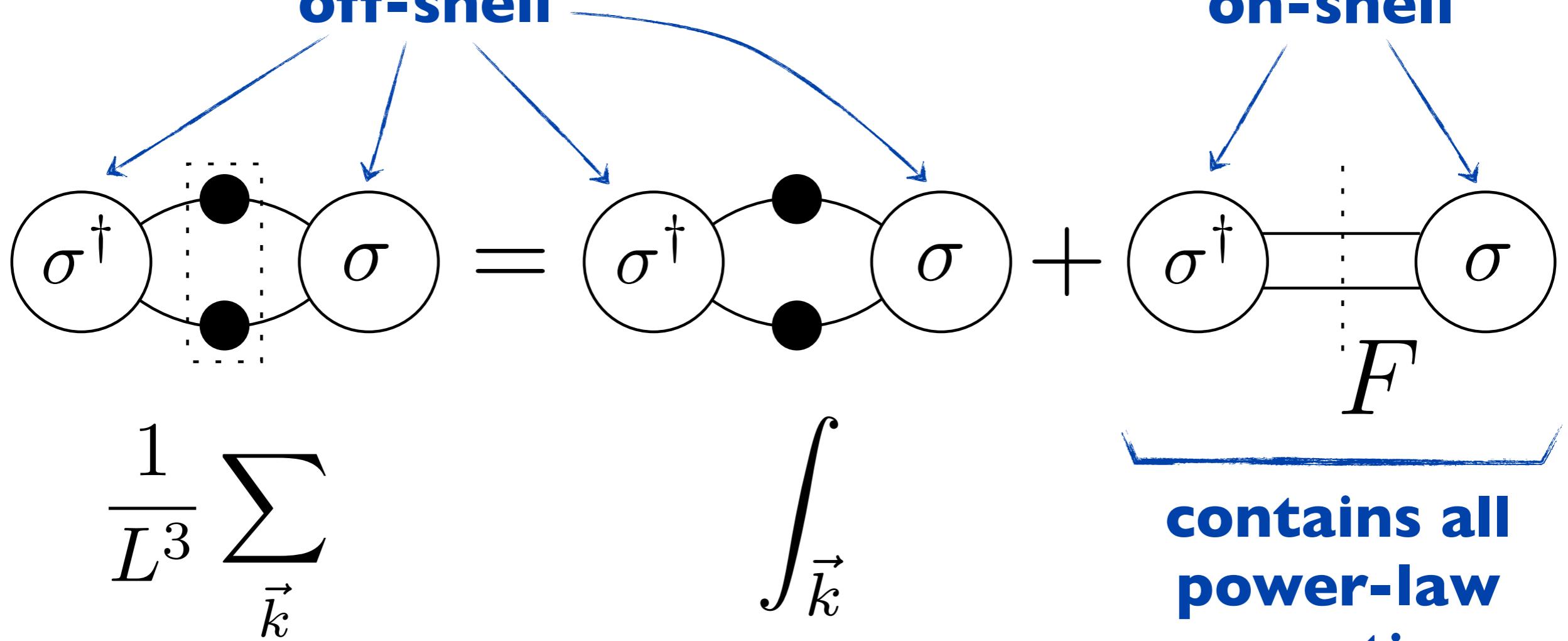
**Next we introduce an important identity**

$$\frac{1}{L^3} \sum_{\vec{k}} \int \vec{k} = \sigma^\dagger \circlearrowleft \sigma + \sigma^\dagger \circlearrowleft F \circlearrowright \sigma$$

**contains all power-law corrections**

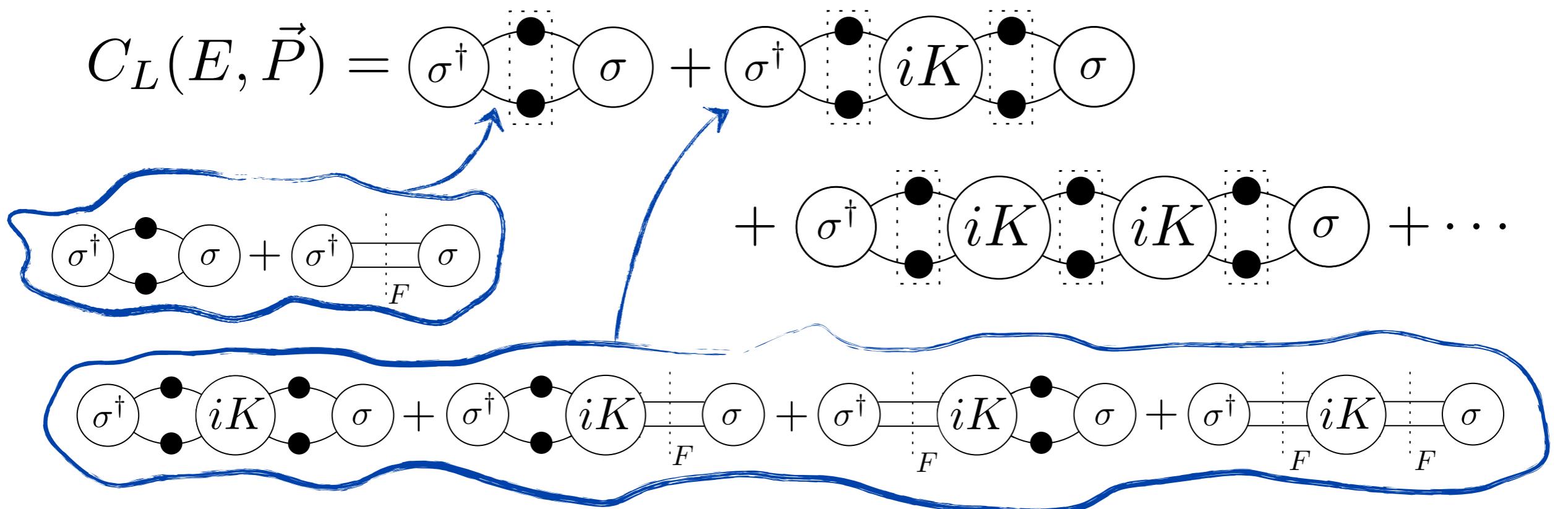
$$C_L(E, \vec{P}) = \sigma^\dagger \text{---} \begin{array}{|c|c|c|c|} \hline & \bullet & \circ & \circ \\ \hline \end{array} + \sigma^\dagger \text{---} \begin{array}{|c|c|c|c|c|c|} \hline & \bullet & \circ & \circ & \bullet & \circ \\ \hline \end{array} iK + \sigma^\dagger \text{---} \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \bullet & \circ & \circ & \bullet & \circ & \bullet & \circ & \circ \\ \hline \end{array} iK + \sigma^\dagger \text{---} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & \bullet & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\ \hline \end{array} iK + \dots$$

Next we introduce an important identity



**contains all  
power-law  
corrections**

$$C_L(E, \vec{P}) = \sigma^\dagger \text{---} \sigma + \sigma^\dagger \text{---} iK \text{---} \sigma + \dots$$



$$C_L(E, \vec{P}) = \sigma^\dagger \circlearrowleft \sigma + \sigma^\dagger \circlearrowleft iK \circlearrowright \sigma + \dots$$

+  $\sigma^\dagger \circlearrowleft iK \circlearrowright iK \circlearrowleft iK \circlearrowright \dots$

$$\begin{aligned} & \sigma^\dagger \circlearrowleft \sigma + \sigma^\dagger \circlearrowleft \sigma \\ & + \sigma^\dagger \circlearrowleft iK \circlearrowright \sigma + \sigma^\dagger \circlearrowleft iK \circlearrowright iK \circlearrowright \sigma \\ & + \sigma^\dagger \circlearrowleft iK \circlearrowright iK \circlearrowleft iK \circlearrowright \sigma + \dots \end{aligned}$$

## Now regroup by number of F cuts

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \underbrace{\left\{ \sigma^\dagger \circlearrowleft \sigma + \sigma^\dagger \circlearrowleft iK \circlearrowright \sigma + \dots \right\}}_F$$

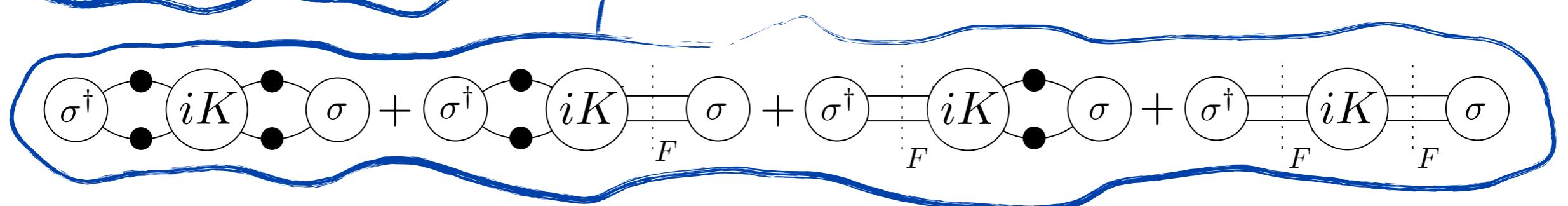
**zero F cuts**

**one F cut**

$A$        $A'$

**these infinite-volume  
quantity do not  
appear in final result**

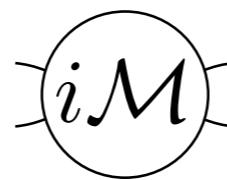
$$C_L(E, \vec{P}) = \sigma^\dagger \sigma + \sigma^\dagger iK \sigma + \dots$$



**Now regroup by number of F cuts**

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A \underset{F}{\cdots} A' + A \underset{F}{\cdots} \left\{ iK + iK \underset{F}{\cdots} iK + \dots \right\} A' + \dots$$

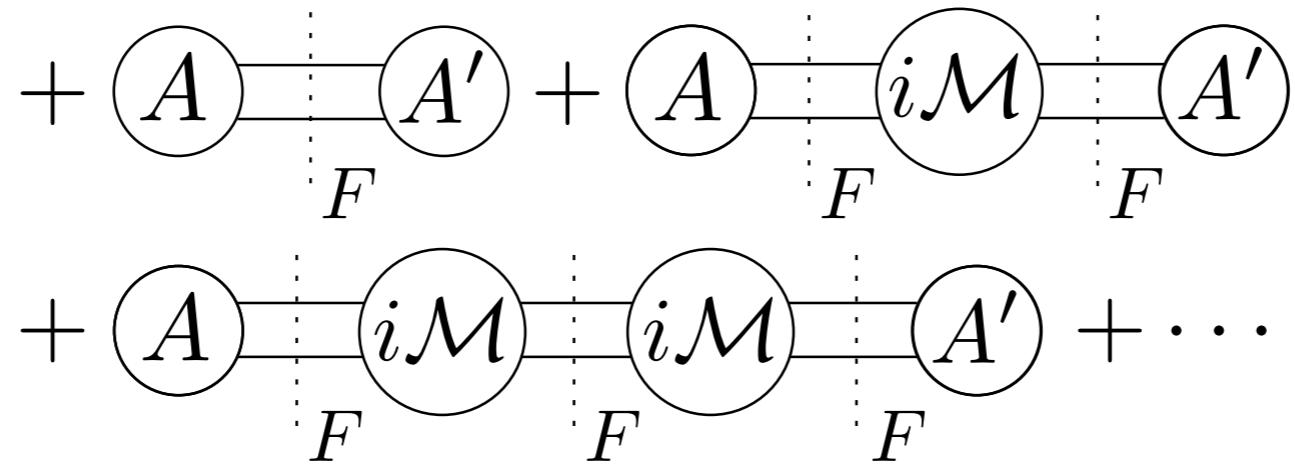
**two F cuts**



**As Promised!**

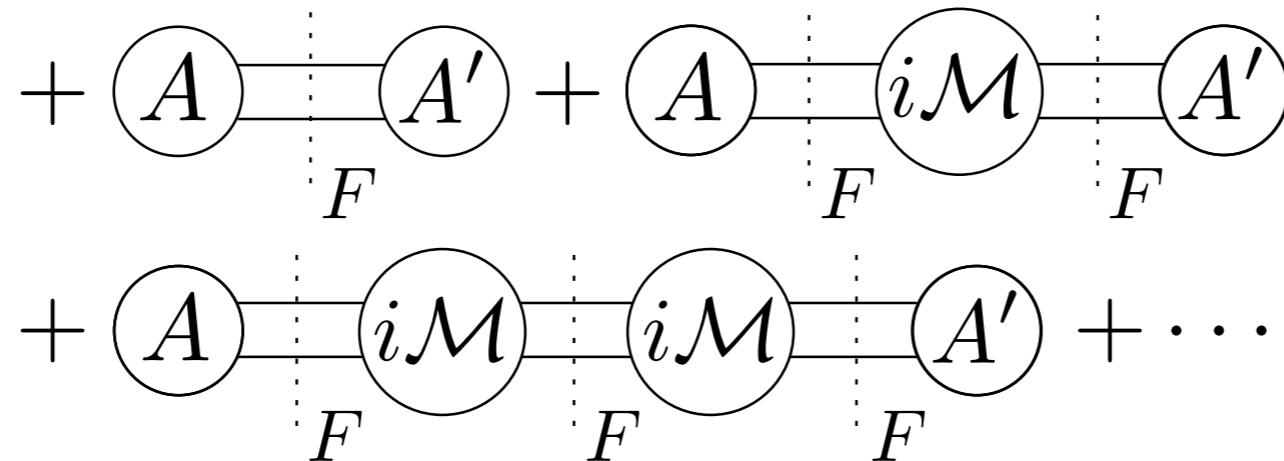
**infinite-volume on-shell two-to-two  
scattering amplitude**

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' i F [i\mathcal{M}_{2 \rightarrow 2} i F]^n A$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' i F [i \mathcal{M}_{2 \rightarrow 2} i F]^n A$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' i F \frac{1}{1 - i \mathcal{M}_{2 \rightarrow 2} i F} A$$

**no poles** **no poles** **no poles**

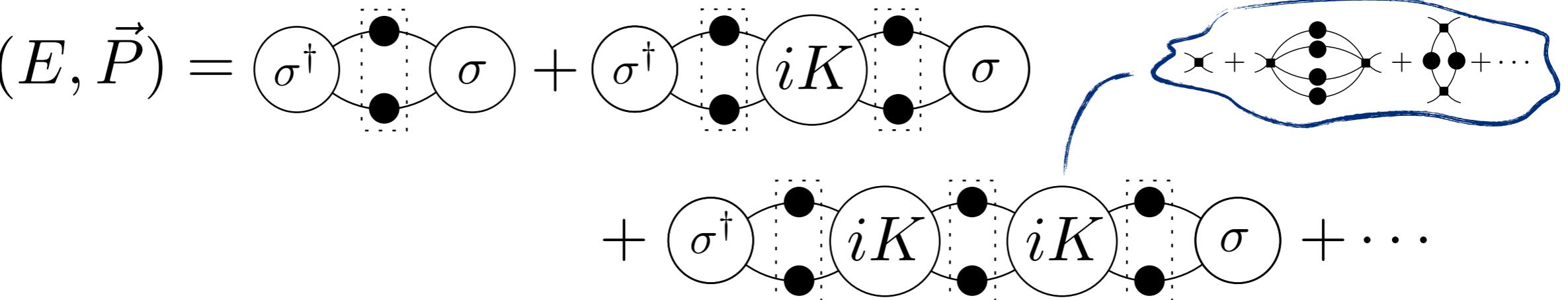
$C_L(E, \vec{P})$  diverges whenever  $i F \frac{1}{1 - i \mathcal{M}_{2 \rightarrow 2} i F}$  diverges

# Two-particle review

1

$$C_L(E, \vec{P}) = \sigma^\dagger \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} \sigma + \sigma^\dagger \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} iK \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} \sigma + \dots$$

+

$$\sigma^\dagger \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} iK \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} iK \text{---} \begin{array}{|c|c|} \hline & \bullet \\ \bullet & \\ \hline \end{array} \text{---} \sigma + \dots$$


# Two-particle review

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \dots$$

Diagram 1: A two-particle vertex function  $C_L$  is shown as a sum of two terms. The first term consists of a  $\sigma^\dagger$  circle connected to a  $\sigma$  circle by two horizontal lines, with a vertical line connecting them. The second term is similar but includes a central  $iK$  circle. Both diagrams are enclosed in a blue wavy bracket labeled '1'.

Diagram 2: A two-particle vertex function  $C_L$  is shown as a sum of two terms. The first term consists of a  $\sigma^\dagger$  circle connected to a  $\sigma$  circle by two horizontal lines, with a vertical line connecting them. The second term is similar but includes a central  $iK$  circle. Both diagrams are enclosed in a blue wavy bracket labeled '2'.

$\dots$

# Two-particle review

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \dots$$

Diagram 1: A series of diagrams showing the expansion of the two-particle correlation function. It starts with a term involving a  $\sigma^\dagger$  and a  $\sigma$  vertex connected by a dashed line. This is followed by a term involving a  $\sigma^\dagger$ , an  $iK$  vertex, and a  $\sigma$  vertex connected by a dashed line. The sequence continues with higher-order terms involving multiple  $iK$  vertices.

Diagram 2: A diagram showing the expansion of the two-particle correlation function starting with a  $\sigma^\dagger$  and a  $\sigma$  vertex connected by a dashed line, plus a term involving a  $\sigma^\dagger$  and a  $\sigma$  vertex connected by a solid line.

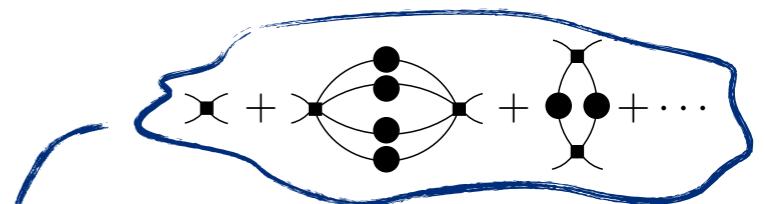
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{Diagram 3} + \dots$$

Diagram 3: A diagram showing the expansion of the two-particle correlation function starting with a  $A$  and an  $A'$  vertex connected by a dashed line, plus a term involving an  $A$  and an  $i\mathcal{M}$  vertex connected by a dashed line, plus a term involving an  $A$  and an  $i\mathcal{M}$  vertex connected by a solid line.

# Two-particle review

$$C_L(E, \vec{P}) = \langle \sigma^\dagger \sigma \rangle + \langle \sigma^\dagger iK \sigma \rangle$$

1



$$\langle \sigma^\dagger \sigma \rangle + \langle \sigma^\dagger \sigma \rangle$$

2

$$+ \langle \sigma^\dagger iK iK \sigma \rangle + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

$$+ \langle A A' \rangle_F + \langle A i\mathcal{M} A' \rangle_F$$

3

$$+ \langle A i\mathcal{M} i\mathcal{M} A' \rangle_F + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

4

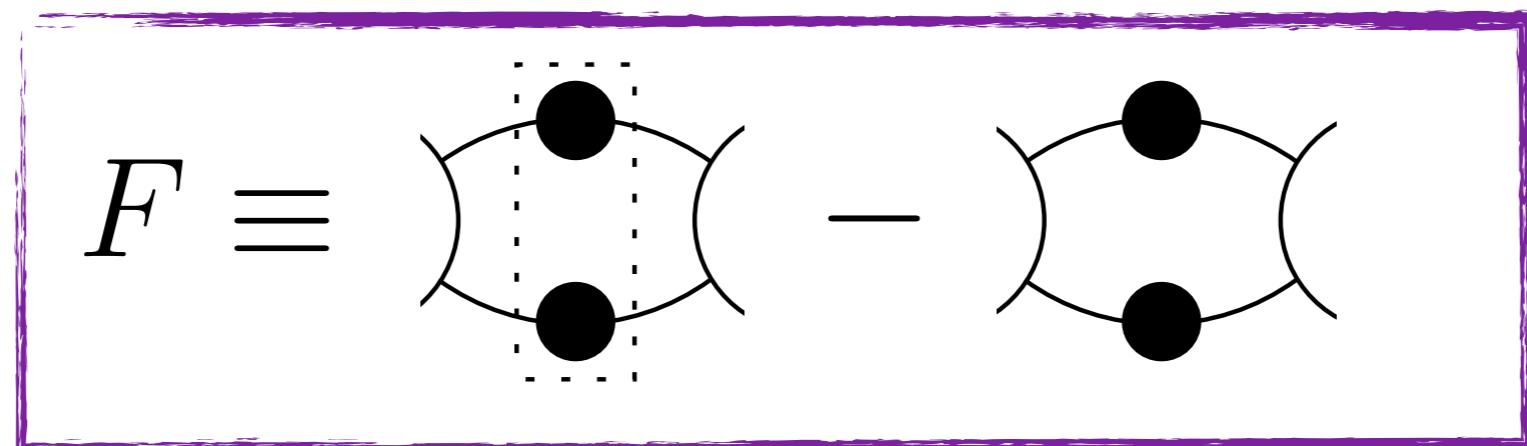
# Two-particle result

At fixed  $(L, \vec{P})$ , finite-volume spectrum  
is all solutions to

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2 \rightarrow 2} iF] = 0$$

diagonal matrix in  
angular momentum space

kinematic  
(related to Lüscher  
Zeta function)



# Two-particle result

$$\Delta_{L,P}(E) = \det[1 - i\mathcal{M}_{2\rightarrow 2}iF] = 0$$

...is it useful?

**At low energies, s-wave dominates**

$$[\mathcal{M}_{2\rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

$$F^s(E, \vec{P}, L) \equiv \frac{1}{2} \left[ \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3} \right] \frac{1}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k} + i\epsilon)}$$

# Two-particle result

**Note also, equation is real**

$$[\mathcal{M}_{2 \rightarrow 2}^s(E_n^*)]^{-1} = -F^s(E_n, \vec{P}, L)$$

$$p_n \cot \delta^s(p_n) - i \cancel{p}_n = -16\pi E_n^* \operatorname{Re} F^s - i \cancel{p}_n$$

This can also be seen by replacing i-epsilon with principal value everywhere in derivation.

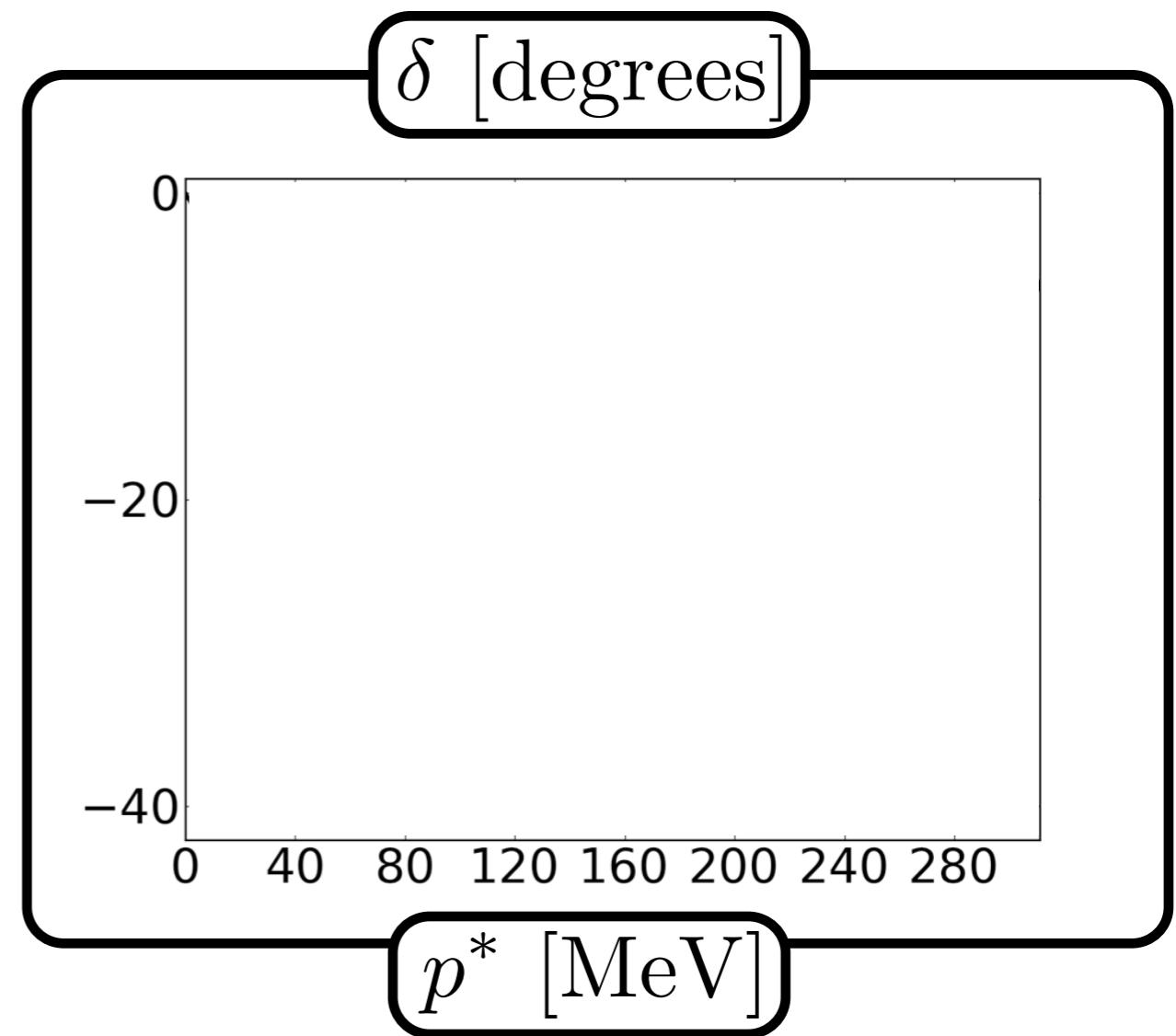
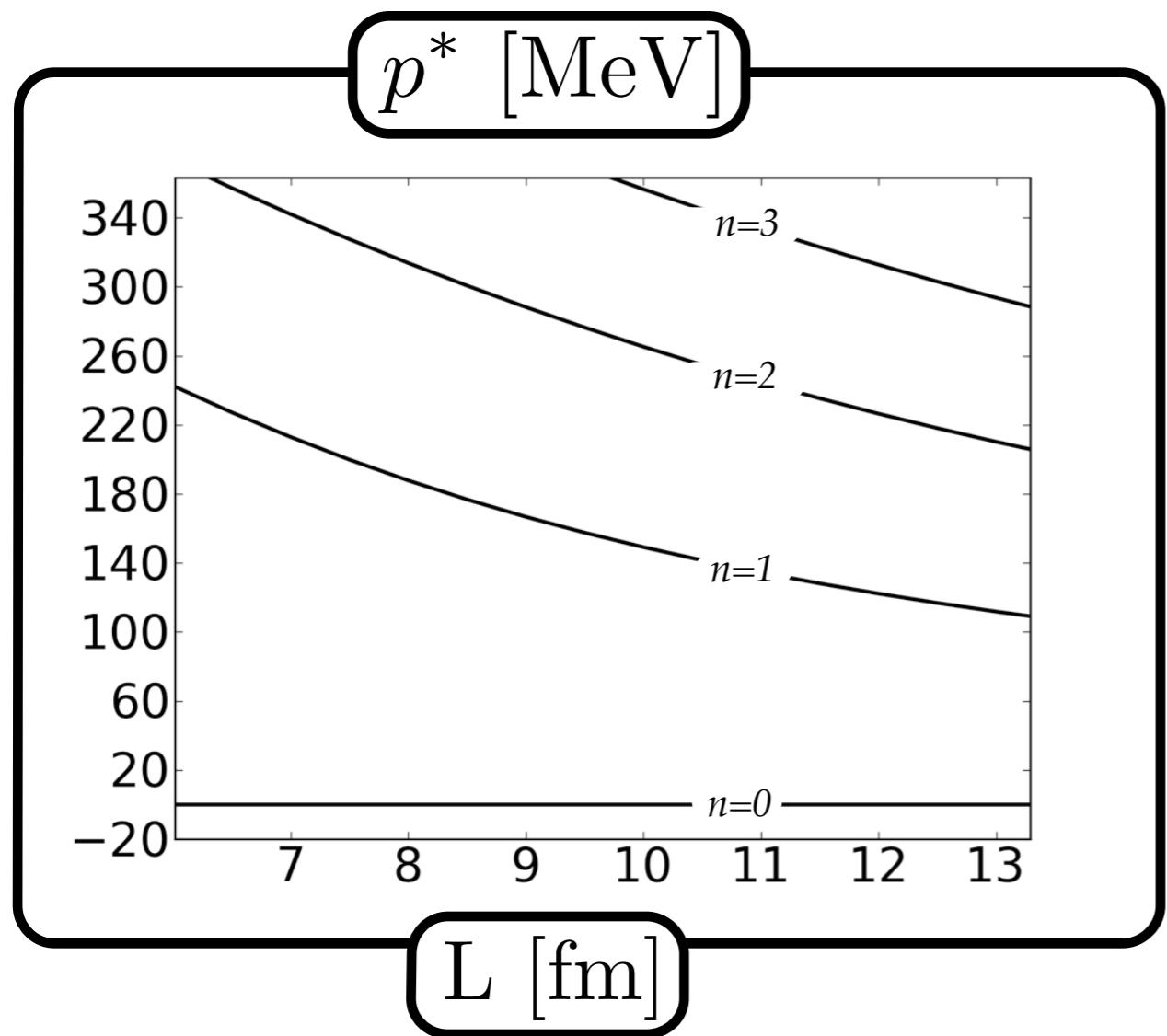
$$\mathcal{M}_{2 \rightarrow 2} \longrightarrow \mathcal{K}_{2 \rightarrow 2}$$

$$F \longrightarrow \operatorname{Re} F$$

**Important for three-particle case**

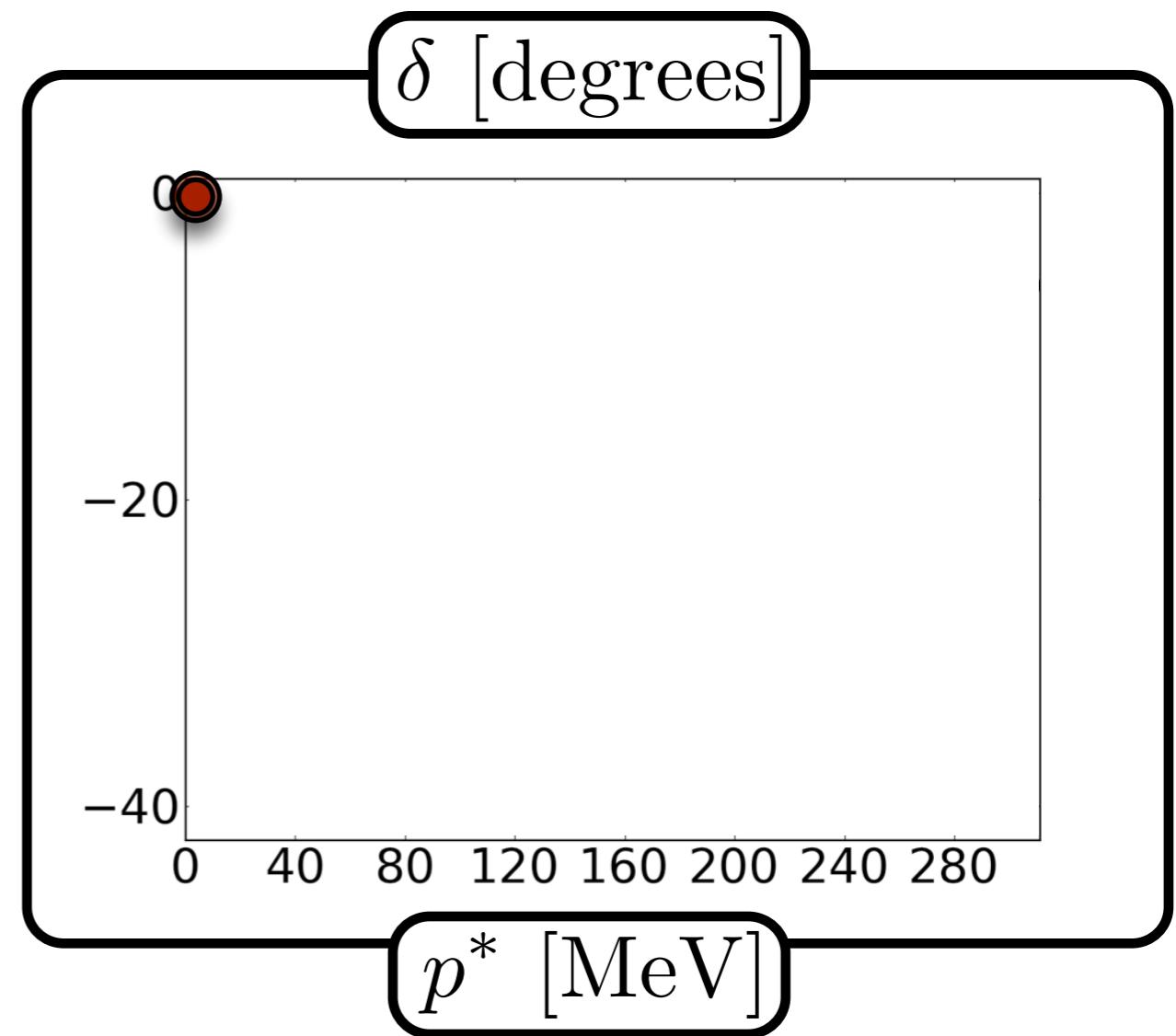
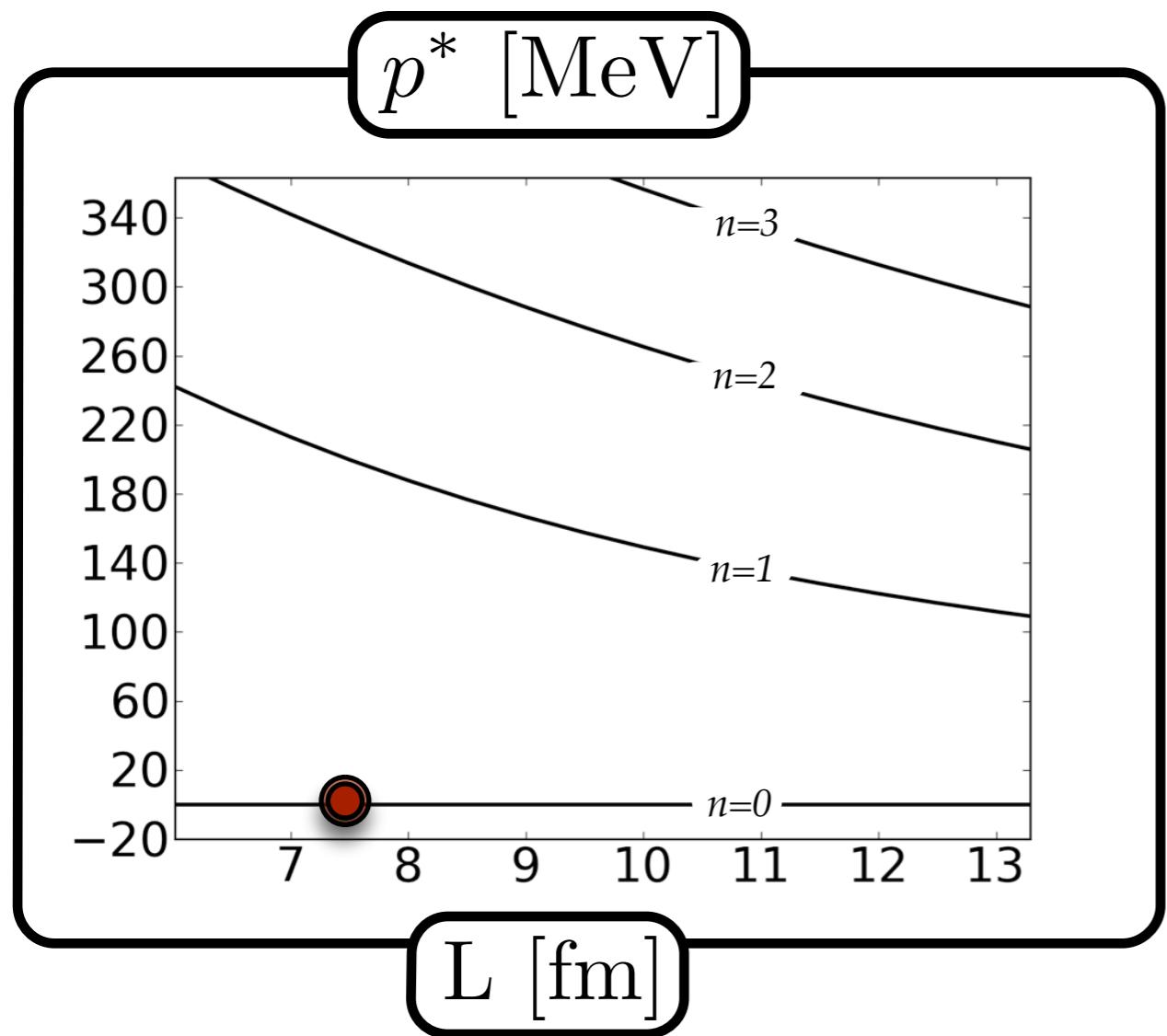
$$p_n \cot \delta(p_n) = -16\pi E_n^* \operatorname{Re} F$$

$$\mathcal{M}_{2 \rightarrow 2}^s(E) = \frac{16\pi E}{p \cot \delta(p) - ip}$$



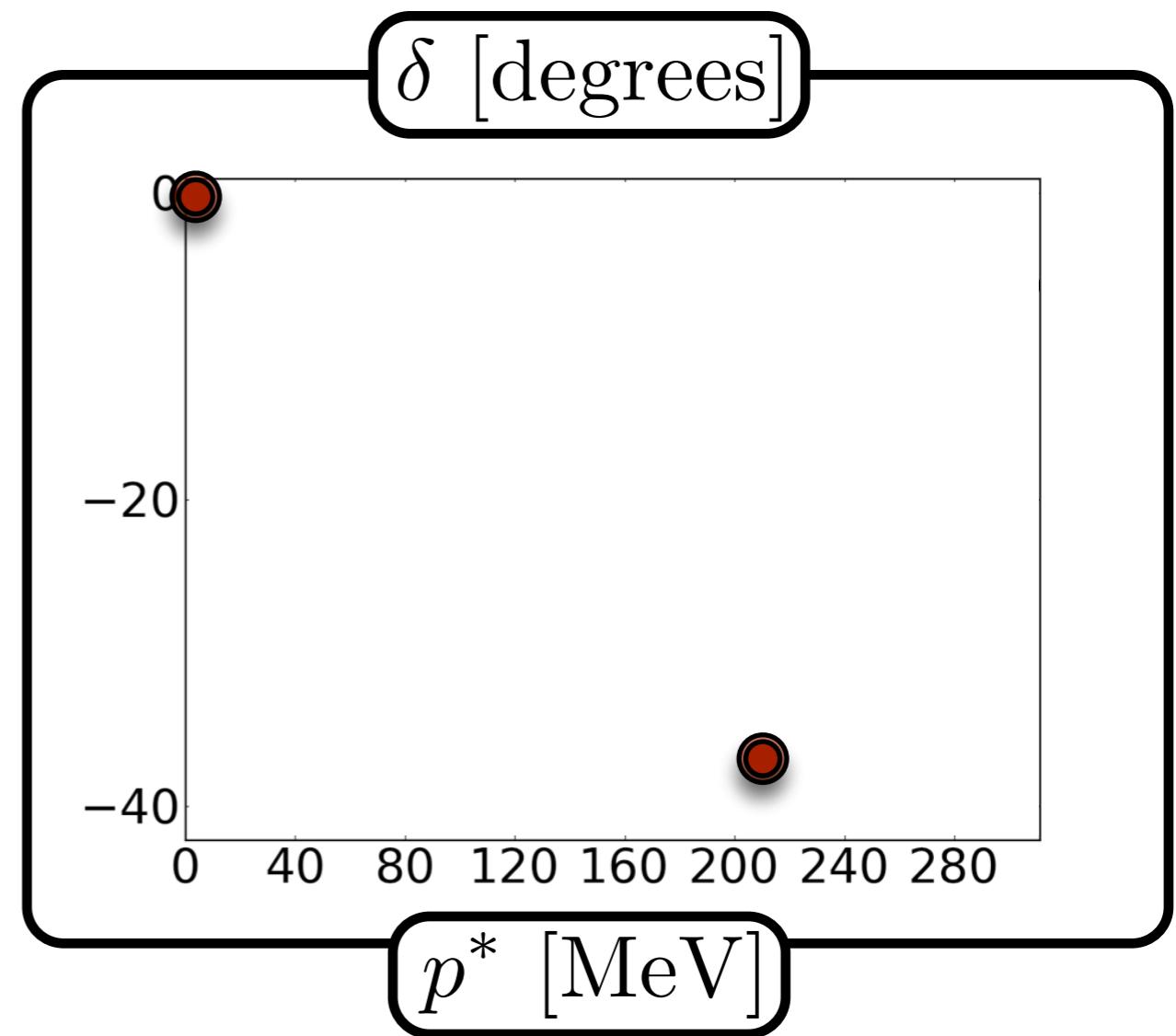
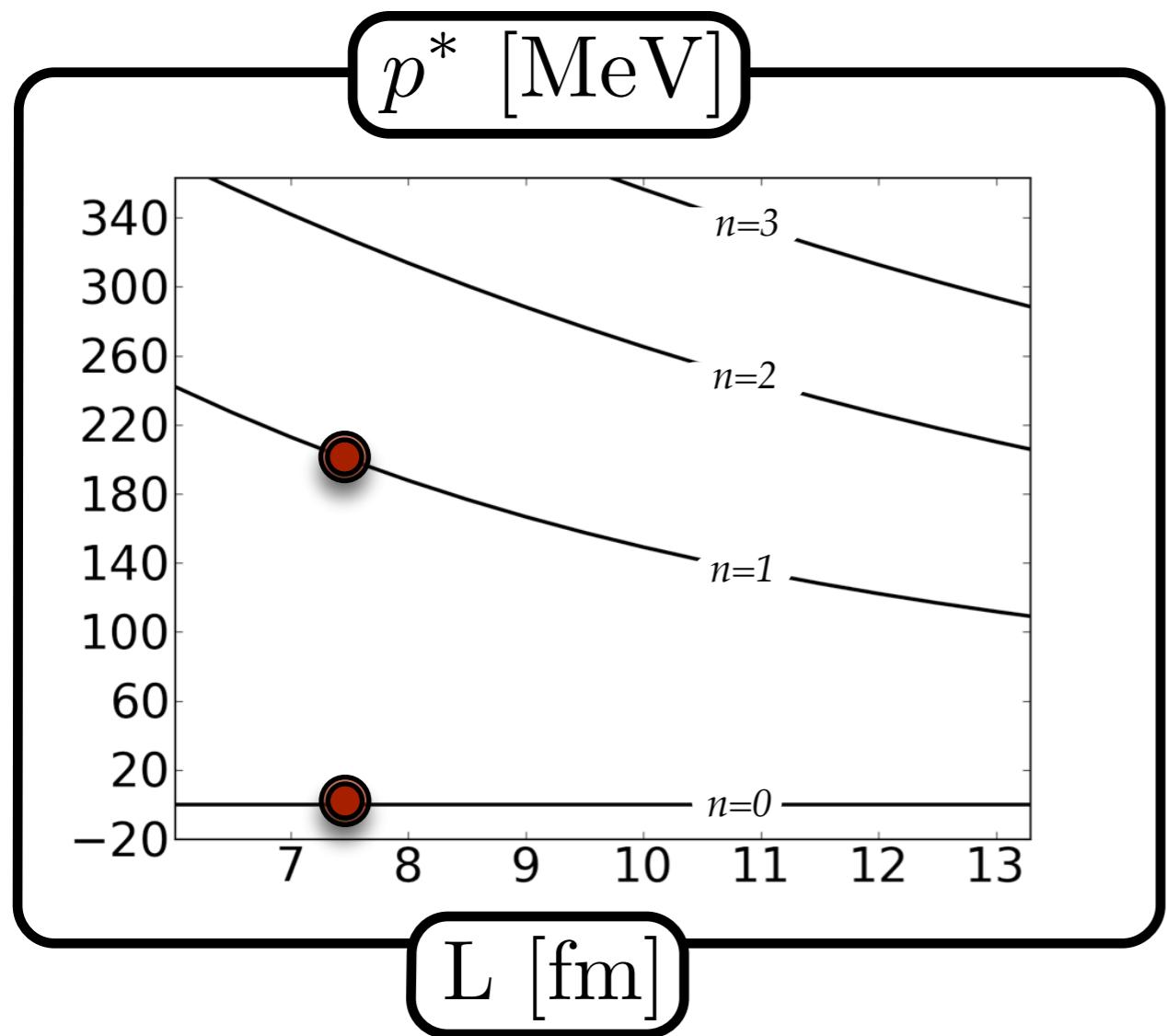
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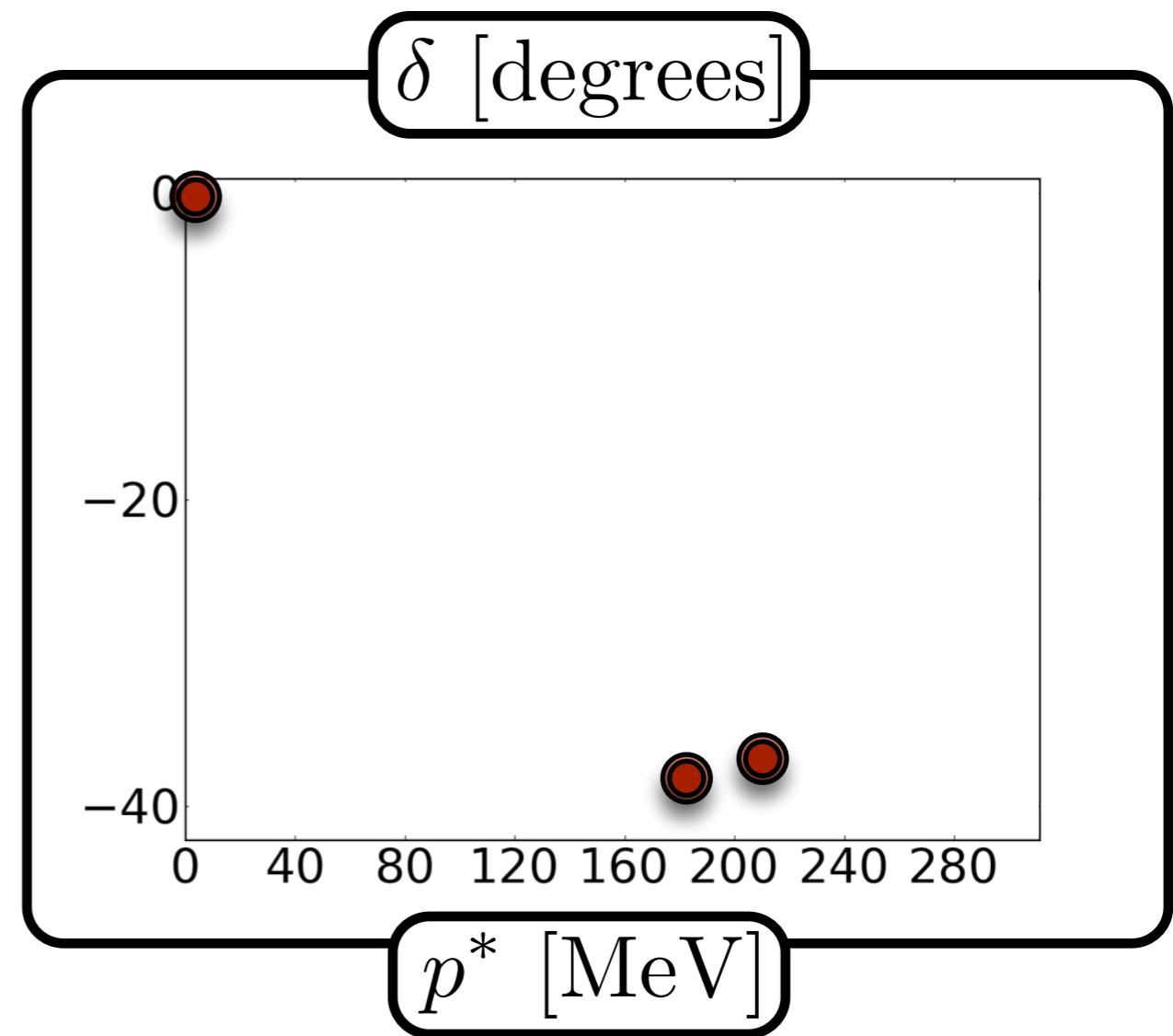
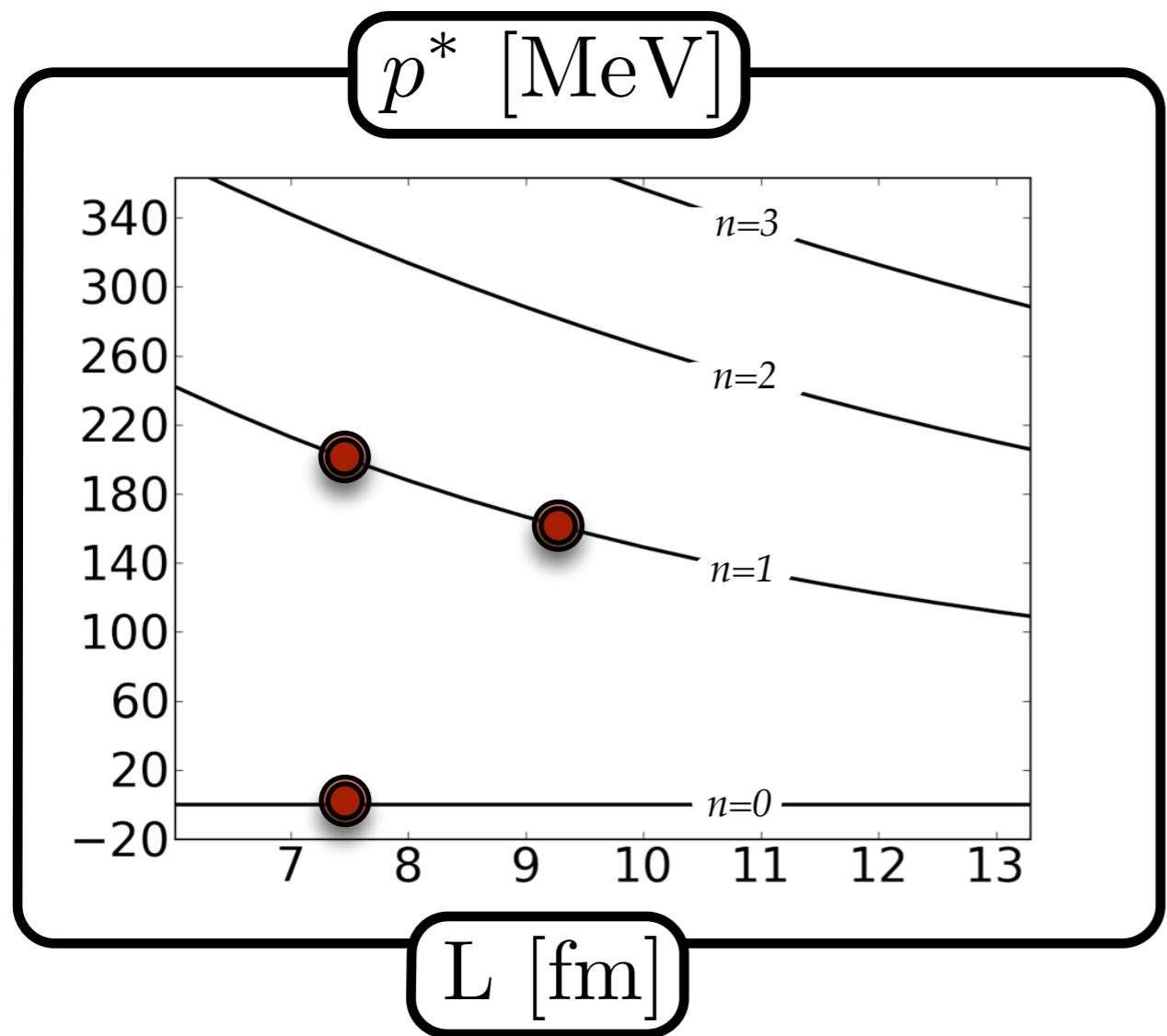
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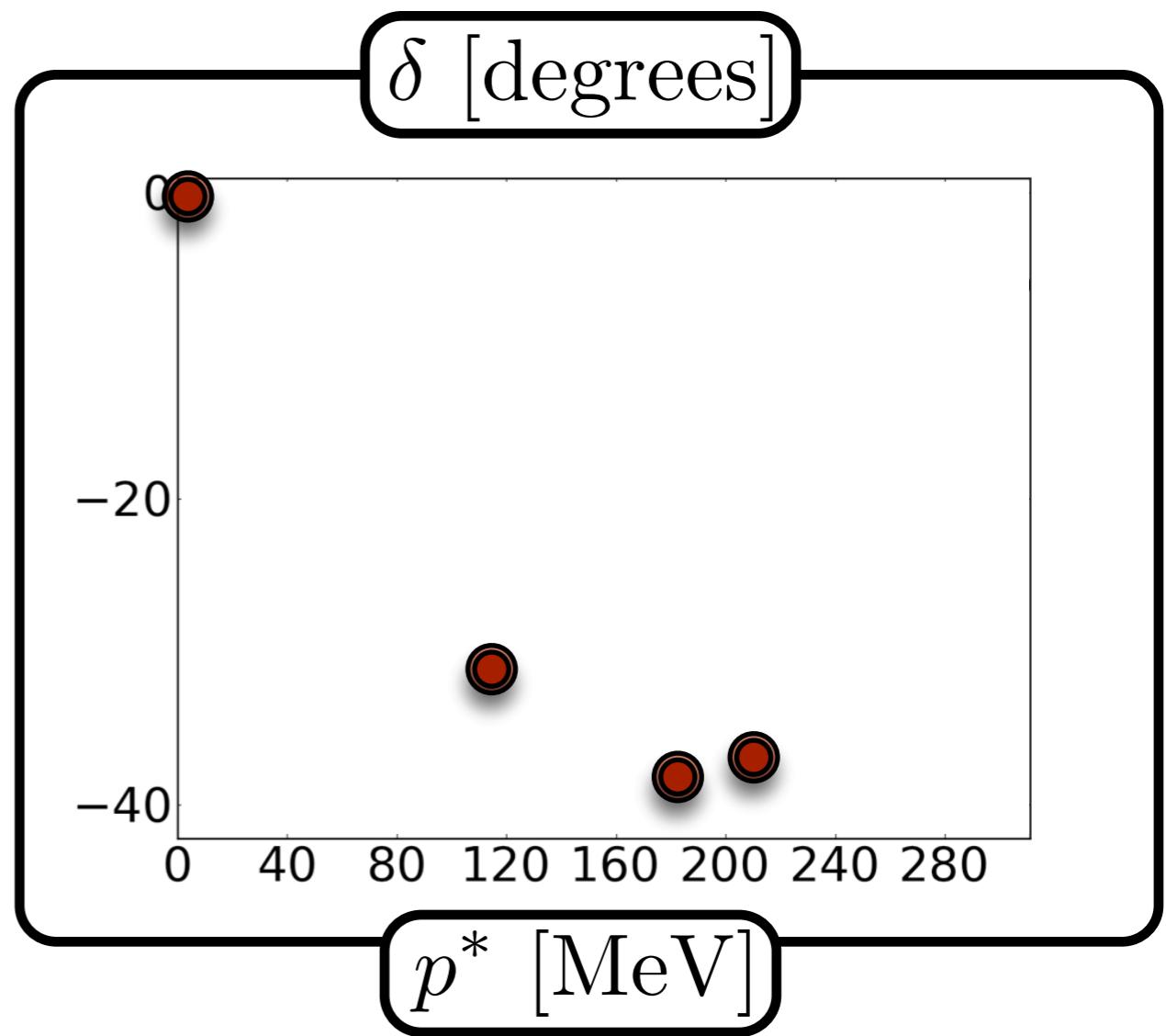
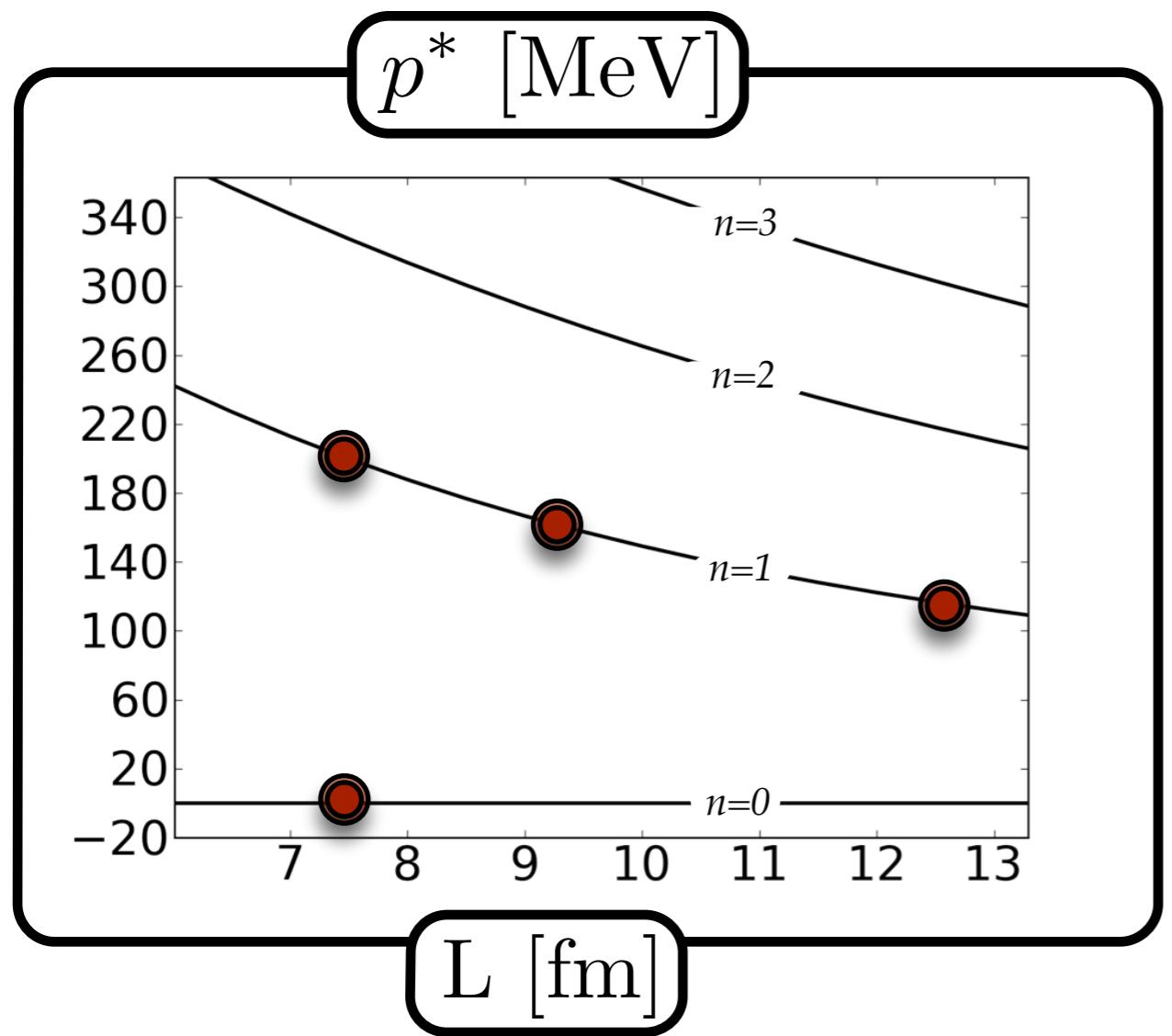
$$p_n \cot \delta(p_n) = -16\pi E_n^* \operatorname{Re} F$$

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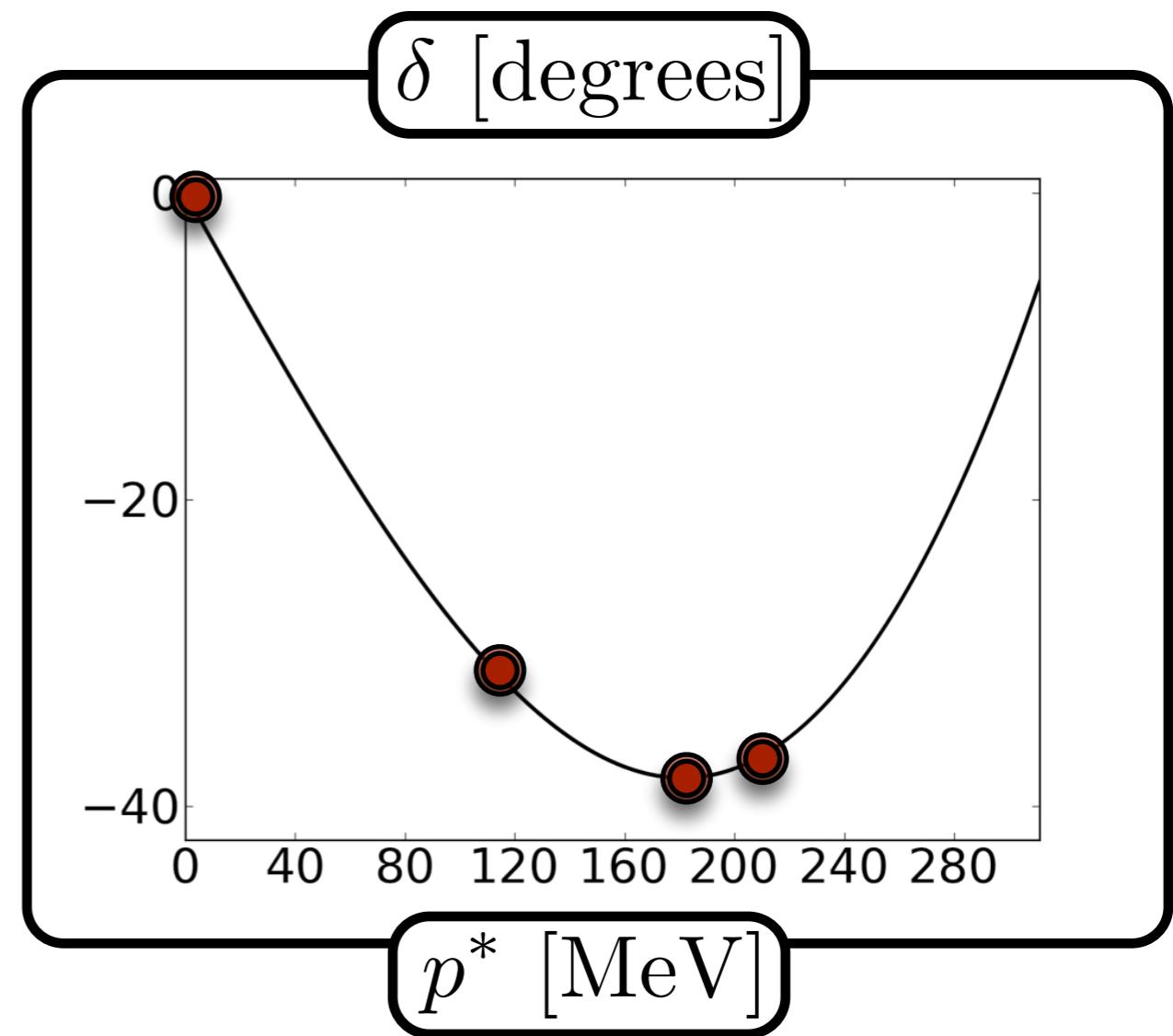
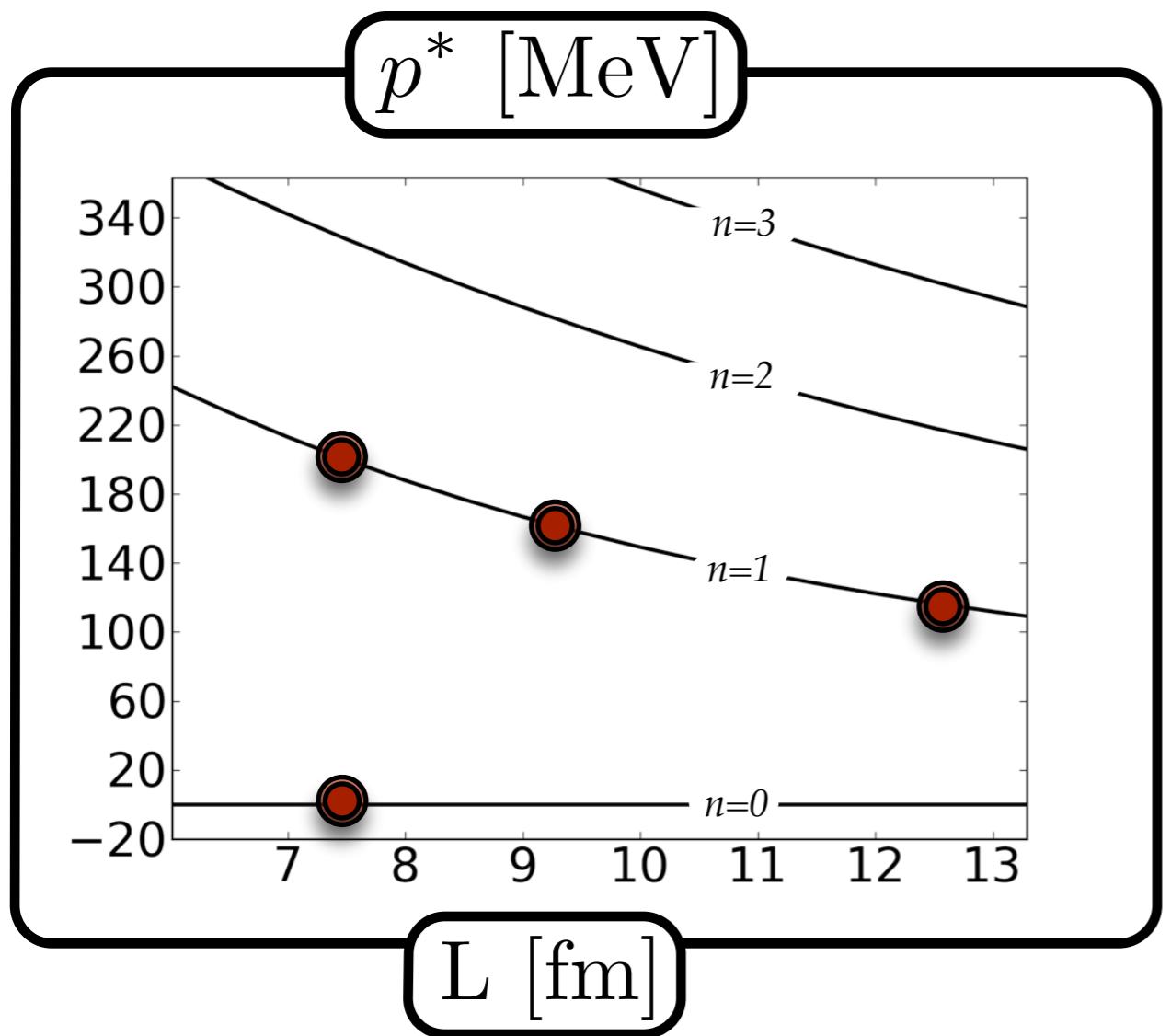
$$p_n \cot \delta(p_n) = -16\pi E_n^* \operatorname{Re} F$$

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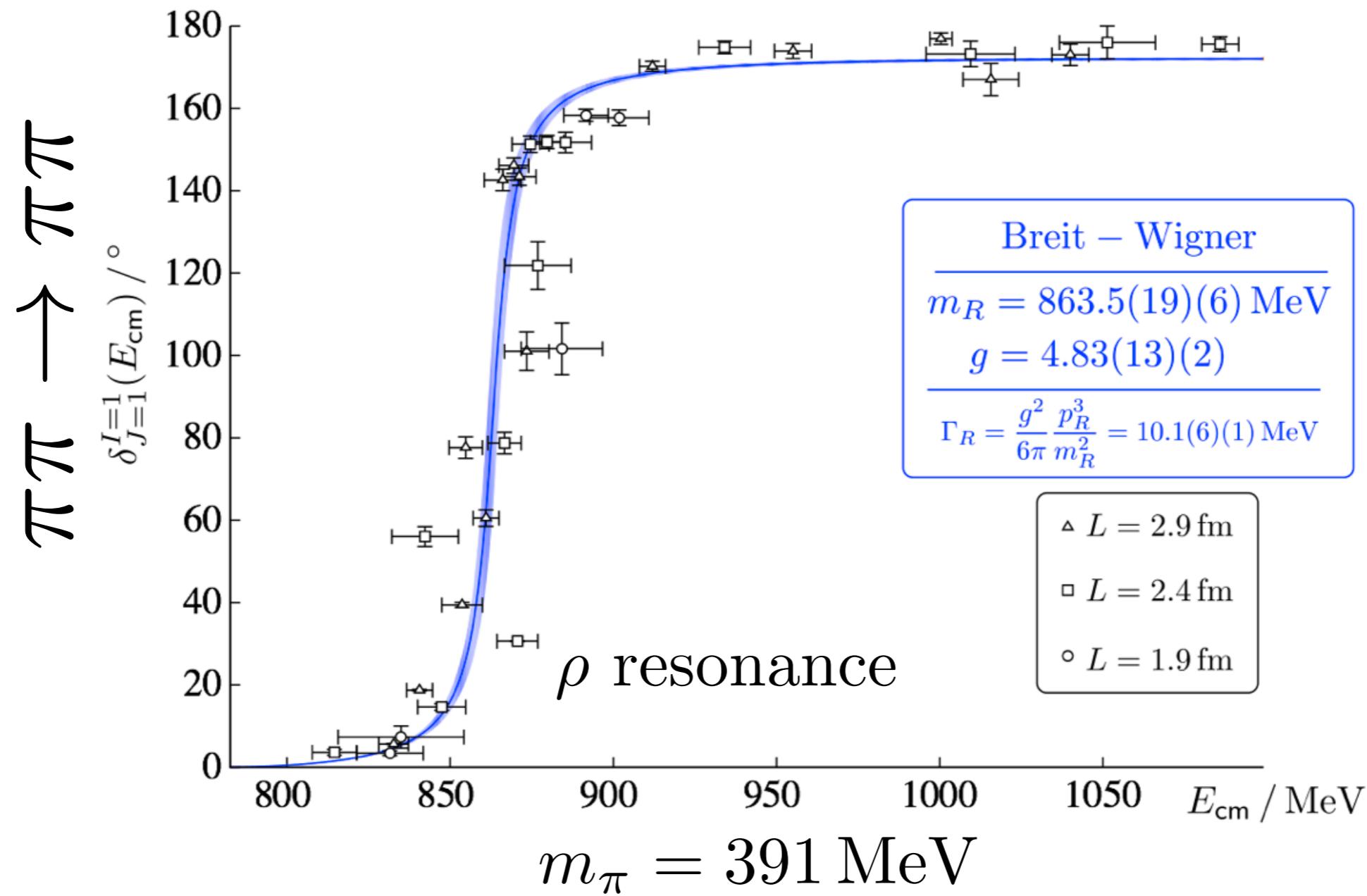


$$p_n \cot \delta(p_n) = -16\pi E_n^* \operatorname{Re} F$$

$$\mathcal{M}_{2 \rightarrow 2}^s(E) = \frac{16\pi E}{p \cot \delta(p) - ip}$$



$$p_n \cot \delta_{J=1}(p_n) = -16\pi E_n^* \operatorname{Re} F_{10;10}(E_n, \vec{P}, L)$$



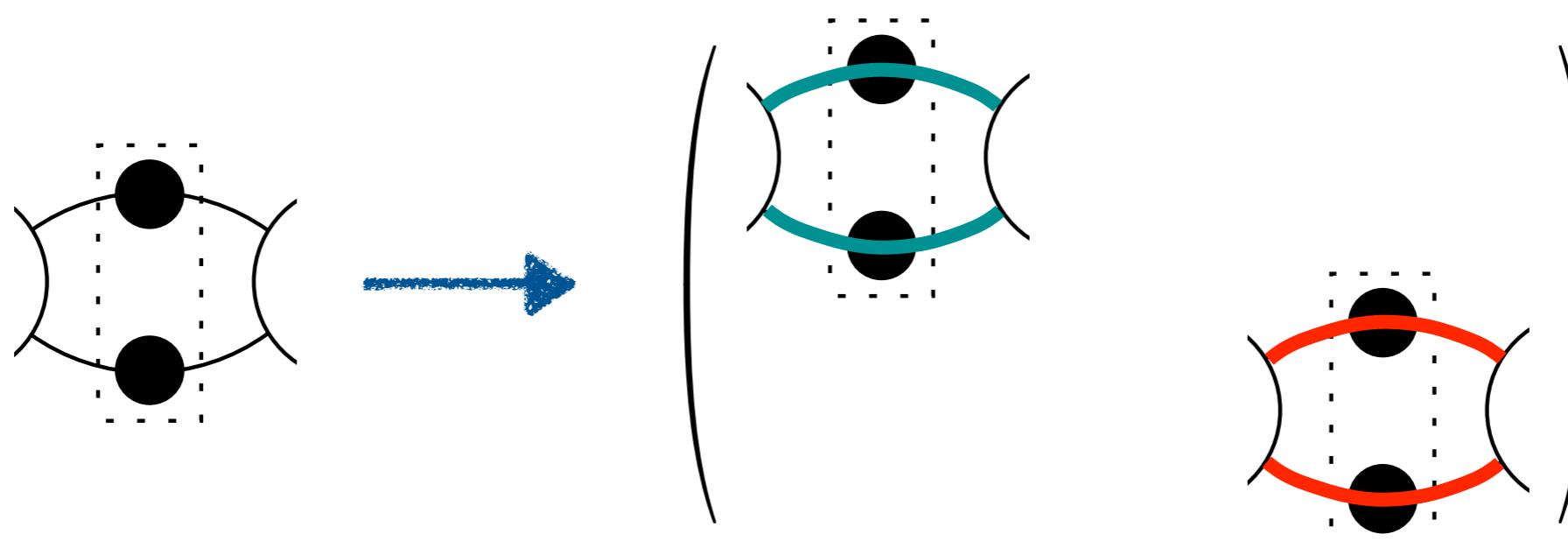
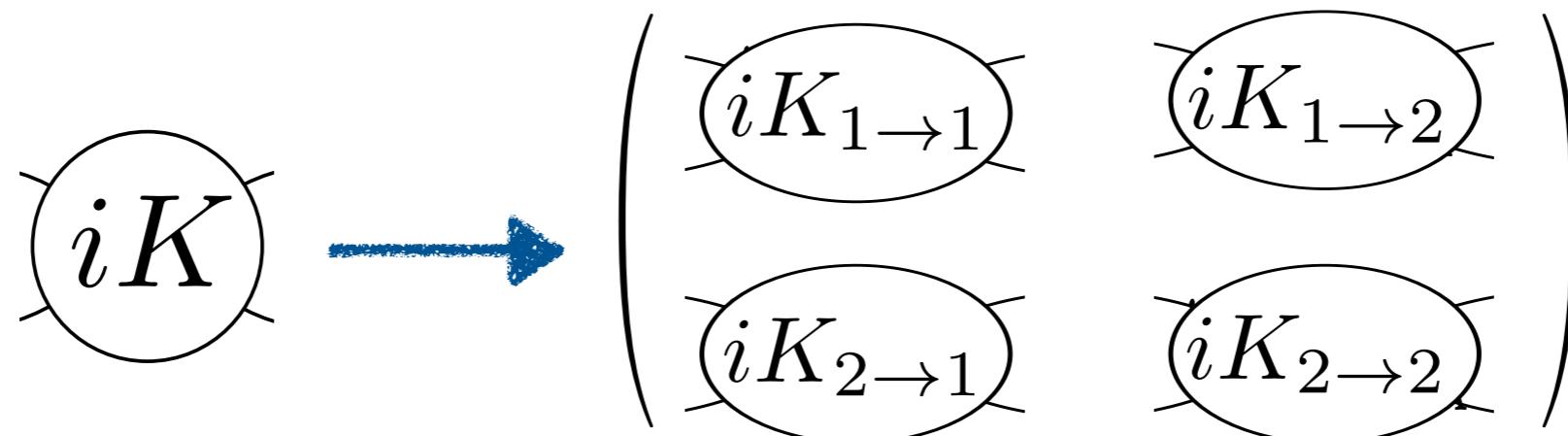
from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

# Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \overline{K}K$$

$$\pi K \rightarrow \eta K$$

Make following replacements



# Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K$$

$$\pi K \rightarrow \eta K$$

One finds

$$\det \left[ 1 - \begin{pmatrix} i\mathcal{M}_{1 \rightarrow 1} & i\mathcal{M}_{1 \rightarrow 2} \\ i\mathcal{M}_{2 \rightarrow 1} & i\mathcal{M}_{2 \rightarrow 2} \end{pmatrix} \begin{pmatrix} iF_1 & 0 \\ 0 & iF_2 \end{pmatrix} \right] = 0$$

M. Lage, U.-G. Meißner, and A. Rusetsky, Phys.Lett., B681, 439 (2009)

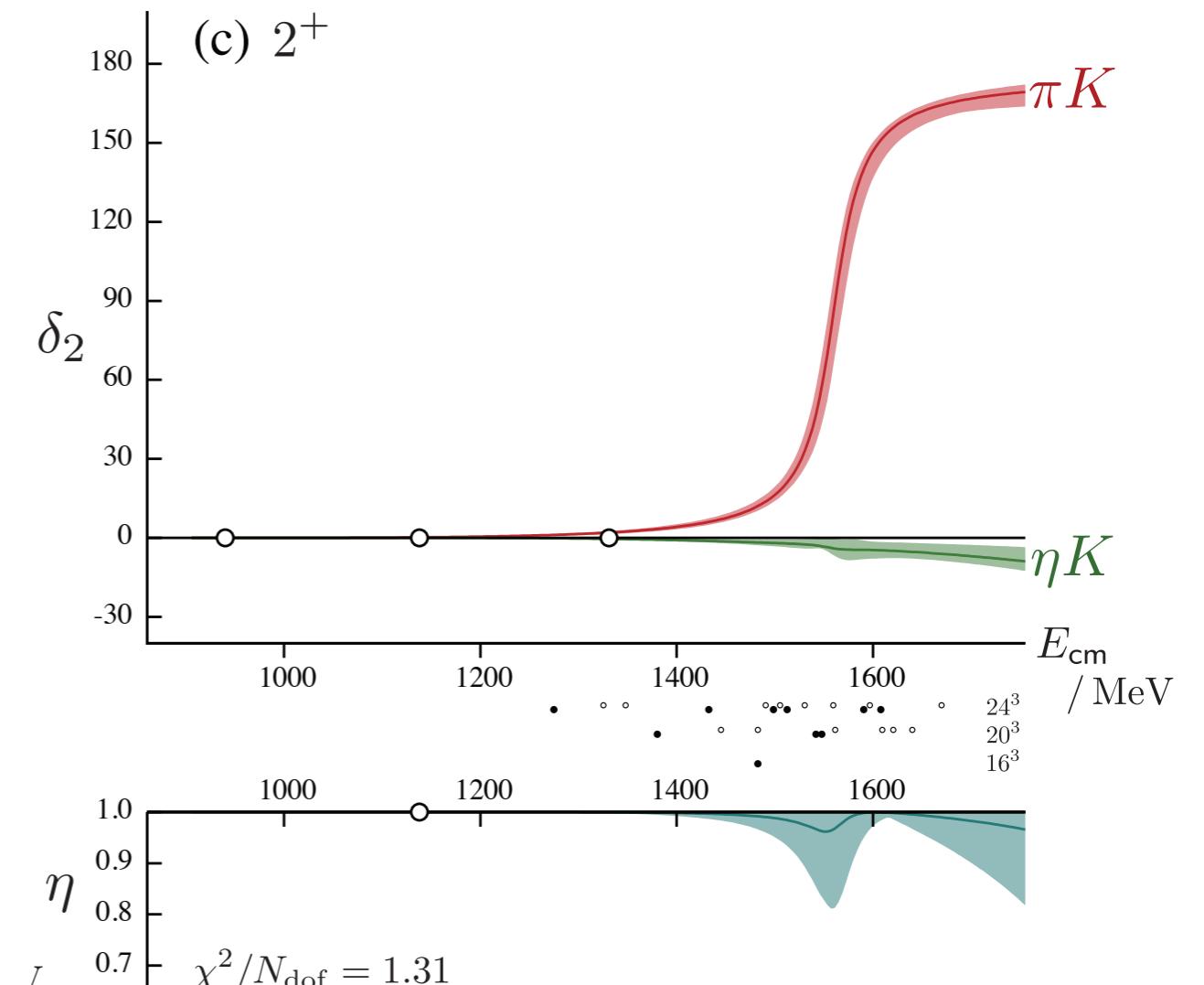
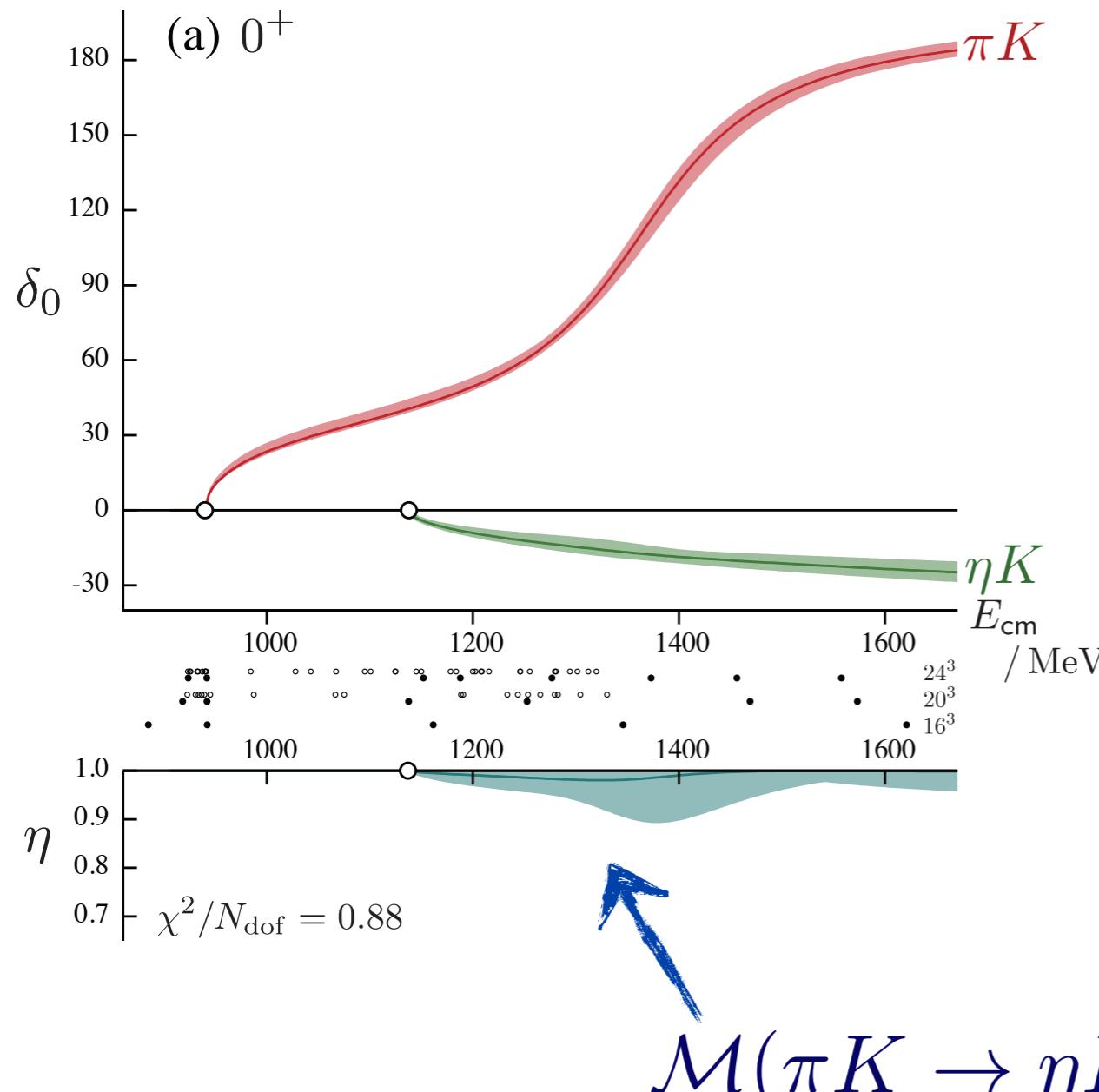
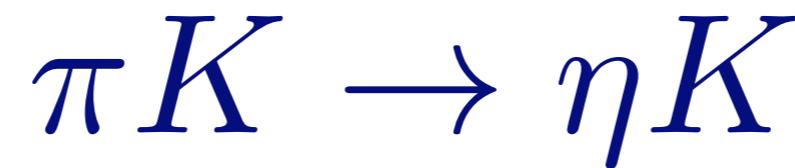
V. Bernard, M. Lage, U.-G. Meißner, and A. Rusetsky, JHEP, 1101, 019 (2011)

M. Döring, U.-G. Meißner, E. Oset, and A. Rusetsky, Eur.Phys.J., A47, 139 (2011)

MTH, S. R. Sharpe, *Phys.Rev. D86* (2012) 016007

R. A. Briceño, Z. Davoudi, *Phys.Rev. D88* (2013) 094507

# Already implemented in LQCD calculation



from Dudek, Edwards, Thomas, Wilson in arXiv:1406:4158